

# SOME CONSIDERATION ON THE SHORT TIME DFT AND ITS APPLICATION TO THE HILBERT TRANSFORMER

## Short Time DFTとHilbert 変換器への 応用に関する研究

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**ABSTRACT** Reducing the spectrum occupancy over radio channels is as well known as important to prevent from exhausting radio spectrum resources. The Hilbert transformer used in SSB or RZ SSB provides with indispensable function to eliminating one side spectrum from modulated signals to efficiently reduce the spectrum occupancy without any distortions(1).

Therefore, many investigations are keenly studied on realizing the Hilbert transformers(2). Unfortunately, most of them have been concerned with approximation in the way of adjusting phase shifting of all pass filters to their arguments being  $\pi/2$  radian with certain amount of phase shifting error. Where these approximated Hilbert transformers are adopted to such SSB radio transmission systems, the approximation error degrades speech quality and expands frequency bandwidth over radio channels.

An exact realization of Hilbert transformers is successfully discussed with employing instantaneous spectrum based on the "short time Discrete Fourier Transform (ab. in ST-DFT)" and is examined to be free from error of phase shifting via computer simulations(3,4).

High speed processing for the ST-DFT Hilbert transformers is also introduced to save great deal of computing power with multi-rate signal processing.

### 1. INTRODUCTION

The ST-DFT Hilbert transform is able to be carried out by shifting the phase of input signals by  $\pi/2$  radian. Where the instantaneous spectrums are given, the ST-DFT Hilbert transform is precisely carried out via merely exchanging real or imaginary part by each other of the instantaneous spectrums.

Hilbert transformed output signals will be

synthesized via inverse short time DFT (ab. in ST-IFT) from the interchanged instantaneous spectrums.

A circuitry configuration of the ST-DFT Hilbert transformer is categorized into three major blocks, namely, I:instantaneous spectrum analyzer, II:frequency domain Hilbert transformer, III:output signal synthesizer. It is easy to understand that the ST-DFT Hilbert transform is ideal and free from any distortion, while the instantaneous spectrum analysis is performed without any errors.

2. INSTANTANEOUS SPECTRUM AND SHORT TIME DFT

The most significant concept in the ST-DFT Hilbert transformer is the instantaneous spectrum. Now, we consider how the short time DFT gives the instantaneous spectrum.

Let the instantaneous spectrum  $\Phi(n)$  at sampling time  $n$  be given by,

$$\Phi(n) = \{\phi_0(n) \ \phi_1(n) \ \phi_2(n) \ \cdots \ \phi_{N-1}(n)\}^T. \quad (1)$$

Where,  $\phi_k(n)$  is a spectrum component at frequency index  $k$  of  $\Phi(n)$ . The spectrum component  $\phi_k(n)$  is defined by the short time DFT as follows,

$$\phi_k(n) = \sum_{r=-\infty}^{\infty} x(r) h(n-r) W_N^{-rk}, \quad (2)$$

here,  $W_N^{-rk} = \exp\{-j(2\pi rk/N)\}$ , integer  $k$  is  $0 \leq k < N$ .

$x(r)$  is an input data at sampling time  $r$ .  $W_N^{-rk}$  is the same operator defined in the existing DFT.

Existence of ST-DFT requires that output signal  $y(n)$  at time  $n$  is precisely produced from the instantaneous spectrum  $\Phi(n)$  via ST-IFT. That is,

$$N \cdot y(n) = \sum_{k=0}^{N-1} \phi_k(n) W_N^{nk},$$

$$W_N^{nk} = \exp\{j(2\pi nk/N)\}. \quad (3)$$

Here,  $W_N^{nk}$  is the same operator as existing inverse DFT.

The ST-IFT denoted by eq.3 requires that the window function  $h(*)$  of eq.2 holds condition  $y(n)=x(n)$ . Substituting eq.2 into eq.3 and exchanging the summation order for variables  $k$  and  $r$ , it gives eq.4.

$$N y(n) = \sum_{k=0}^{N-1} \{ \sum_{r=-\infty}^{\infty} x(r) h(n-r) W_N^{-rk} \} W_N^{nk}$$

$$= \sum_{r=-\infty}^{\infty} x(r) h(n-r) \{ \sum_{k=0}^{N-1} W_N^{(n-r)k} \} \quad (4)$$

The summation for variable  $k$  takes non zero value by  $N$ , only if  $n-r=2Nq$ . Here,  $q$  is integer. This gives a restrict condition to the window function as follows,

$$h(p) = \begin{cases} 1, & \text{if } p = 0 \\ 0, & \text{if } p = 2Nu, \end{cases} \quad (5)$$

u is non zero integer.

For example, an  $N$  frame length Nyquist window function truncated with  $2m$  frame number  $h(p)$ ,

$$h(p) = \sin(p\pi/N) / (p\pi/N), \quad -mN \leq p \leq mN, \quad (6)$$

are able to satisfy eq.5. Hereafter, the truncated Nyquist will be employed for the present as a window function in the ST-DFT.

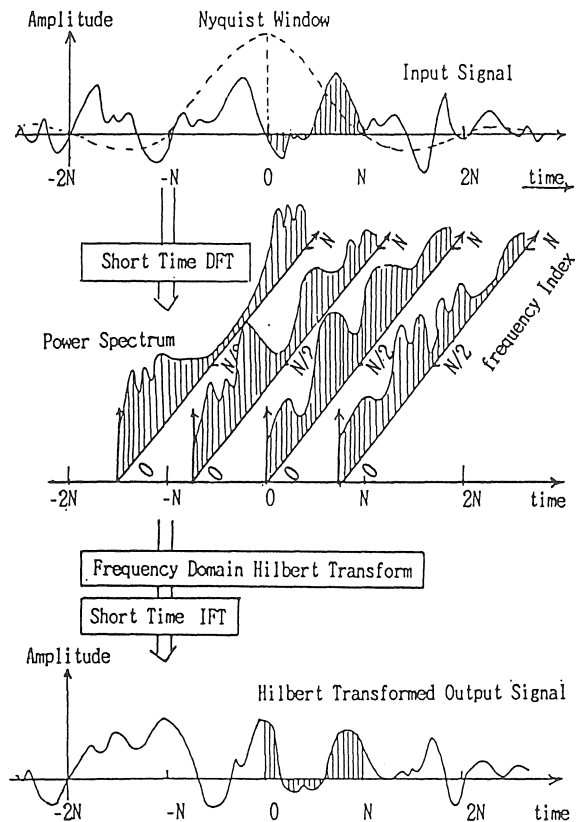


Fig.1 Processing Outline in the ST-DFT Hilbert Transform

### 3. ST-DFT HILBERT TRANSFORMER AND ITS OPERATORS

The processing outline in the ST-DFT Hilbert transformer is shown in fig.1. In the ST-DFT Hilbert transformer, input signal  $x(r)$  is at first analyzed by ST-DFT to yield instantaneous spectrum  $\Phi(n)$ . Secondly, both real and imaginary parts of each component of the instantaneous spectrum are exchanged with each other to get Hilbert transformed spectrum  $\widehat{\Phi}(n)$ . Output signal  $\widehat{y}(n)$  is finally synthesized through ST-IFT from the transformed instantaneous spectrum.

The two processing of instantaneous spectrum analysis and phase shifting being combined into one operator, the frequency domain Hilbert transform operator  $\widehat{W}_N^{-rk}$  is given as follows,

$$\widehat{W}_N^{-rk} = \begin{cases} \exp\{-j(2\pi rk/N + \pi/2)\}, & \text{if } 0 < k < N/2 \\ 0, & \text{if } k = 0, N/2 \\ \exp\{-j(2\pi rk/N - \pi/2)\}, & \text{if } N/2 < k < N. \end{cases}$$

Here,  $j$  is complex unit,  $j = \sqrt{-1}$ . (7)

**Proof: Existence of Frequency Domain**

*Hilbert Transform Operator*

Constant  $\pi/2$  on the first row of eq.7 gives features of phase shifting function by  $\pi/2$  radian to the  $k$ -th component  $\phi_k(n)$  of  $\Phi(n)$ .

This also clearly shows that the frequency domain Hilbert transform is free from amplitude distortion, because operator  $\widehat{W}_N^{-rk}$  consists of single complex exponential function only with pure imaginary variables.

Both the second and third rows of eq.7 satisfy that  $\widehat{y}(n)$  exists physically. Substituting  $\widehat{\phi}_k(n)$  with  $\phi_k(n)$  of eq.3,  $\widehat{y}(n)$  is given as follows,

$$\begin{aligned} N \cdot \widehat{y}(n) &= \sum_{k=0}^{N-1} \widehat{\phi}_k(n) W_N^{nk} \\ &= \widehat{\phi}_0(n) W_N^0 + \widehat{\phi}_{N/2}(n) W_N^{nN/2} \\ &\quad + \sum_{k=1}^{N/2-1} \{ \widehat{\phi}_k(n) W_N^{nk} + \widehat{\phi}_{N-k}(n) W_N^{n(N-k)} \} \end{aligned} \quad (8)$$

In eq.8,  $W_N^0$  or  $W_N^{nN/2}$  takes pure real 1 or  $1(-1)$ , respectively. If  $\phi_0(n)$  or  $\phi_{N/2}(n)$  is non zero, the corresponding transformed component  $\widehat{\phi}_0(n)$  or  $\widehat{\phi}_{N/2}(n)$  takes pure imaginary value and the output  $\widehat{y}(n)$  diversifies to complex. So  $\widehat{W}_N^{-rk}$  have to vanish at  $k=0$  and  $N/2$  as defined in eq.7.

Since  $W_N^{n(N-k)}$  is given by complex conjugate with  $W_N^{nk}$ , i.e.  $W_N^{n(N-k)} = \overline{W_N^{nk}}$ , every bracketted term of eq.8 takes a pure real number, iff  $\widehat{\phi}_{N-k}(n)$  is complex conjugate with  $\widehat{\phi}_k(n)$ . In practice,  $\widehat{\phi}_{N-k}(n)$  is given as

$$\begin{aligned} \widehat{\phi}_{N-k}(n) &= \sum_{r=-\infty}^{\infty} x(r)h(n-r) \exp\{-j\{2\pi r(N-k)/N - \pi/2\}\} \\ &= \sum_{r=-\infty}^{\infty} x(r)h(n-r) \exp\{j\{2\pi rk/N + \pi/2\}\} \\ &= \overline{\phi_k(n)}. \end{aligned} \quad (9)$$

Therefore, the output  $\widehat{y}(n)$  becomes pure real number as follows,

$$\begin{aligned} N \cdot \widehat{y}(n) &= \sum_{k=1}^{N/2-1} \{ \widehat{\phi}_k(n) W_N^{nk} + \widehat{\phi}_{N-k}(n) W_N^{n(N-k)} \} \\ &= \sum_{k=1}^{N/2-1} \{ \widehat{\phi}_k(n) W_N^{nk} + \overline{\phi_k(n)} \overline{W_N^{nk}} \} \\ &= \sum_{k=1}^{N/2-1} 2\text{Real} = \{ \widehat{\phi}_k(n) W_N^{nk} \}, \quad \text{QED.} \end{aligned} \quad (10)$$

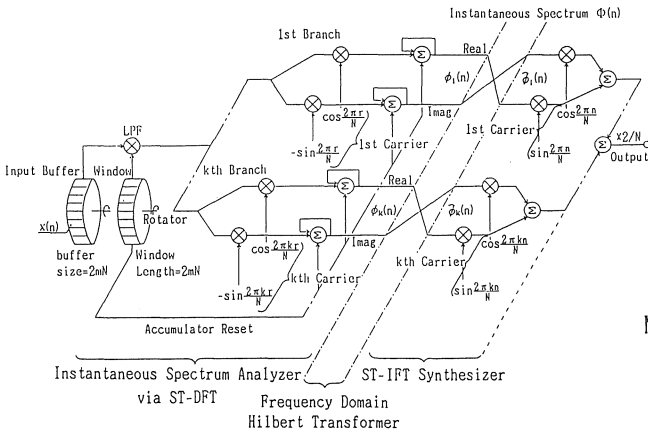


Fig.2 Configuration of the ST-DFT Hilbert Transformer

4. CIRCUITRY CONFIGURATION

The ST-DFT Hilbert transformer consists of three major blocks as shown in fig.2. The first block is the ST-DFT analyzer and consists of  $N/2-1$  modules in which every component  $\phi_k(n)$  is yielded. Inner product of  $x(n)$  and  $W_N^{-rk}$  in eq.2 is performed of modulating the input  $x(n)$  with complex carrier  $W_N^{-rk}$  of  $2\pi k/N$  normalized angular frequency. Convolution  $=\{x(r) W_N^{-rk}\}$  and  $h(r)$  in eq.2 is also interpreted as low-pass filtering the modulated signal  $\{x(r) W_N^{-rk}\}$  by  $h(r)$ . The second block is a Hilbert transformer on the frequency domain. This block is dominant in function, however, it is so simply implemented as two crossing wires to exchange the real with the imaginary part of  $\phi_k(n)$ . The first and second blocks are practically combined together in frequency index wise to get  $\hat{\phi}_k(n)$  directly by adopting  $\hat{W}_N^{-rk}$  instead of  $W_N^{-rk}$  during the modulation. The last is a ST-IFT synthesizer to produce time domain Hilbert transformed signals. In similar to the first block, ST-IFT synthesizer is performed of modulating Hilbert transformed spectrum component  $\hat{\phi}_k(n)$  with complex carrier  $W_N^{rk}$ .

The unit sample response  $I_s(n)$ , shown in fig.3, of the ST-DFT Hilbert transformer is given as follows.

$$I_s(n) = \begin{cases} \frac{2}{N} \frac{\sin(2\pi n/N)}{\{1-\cos(2\pi n/N)\}} \frac{\sin(\pi n/N)}{\pi n/N} \\ = 2\cos(\pi n/N)/n\pi, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases} \quad (11)$$

Equation 12 gives the unit sample response  $I_m(n)$  of the Rabinor's minmax FIR Hilbert (ab. in minmax) transformer(2).

$$I_m(n) = \begin{cases} 2\sin^2(\pi n/2)/n\pi = \{1-\cos(\pi n)\} / n\pi \\ = 2/\pi n, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases} \quad (12)$$

It is shown in both eqs.11 and 12 that the ST-DFT Hilbert transformer enhances minmax transformer. That is,

$$\lim_{N \rightarrow \infty} I_s(n) = \lim_{N \rightarrow \infty} \frac{2\cos(\pi n/N)}{n\pi} = \frac{2}{\pi n} = I_m(n) \quad (13)$$

Here,  $n$  is odd.

5. SPEEDING IN PROCESSING

Literal processing based on eqs.2, 3 and 7 requires a great deal of computing power through ST-DFT Hilbert transform. The Hilbert transformed output signals  $\hat{y}(n)$  are synthesized from interpolated instantaneous spectrum component  $\hat{\phi}_k(n)$  reproduced from  $\hat{\phi}_k(r)$  at every  $R$  sampling as follows.

$$N\hat{y}(n) = \sum_{k=0}^{N-1} \{ \sum_{r=L}^R f(n-rR) \hat{\phi}_k(r) \} W_N^{nk}, \quad (14)$$

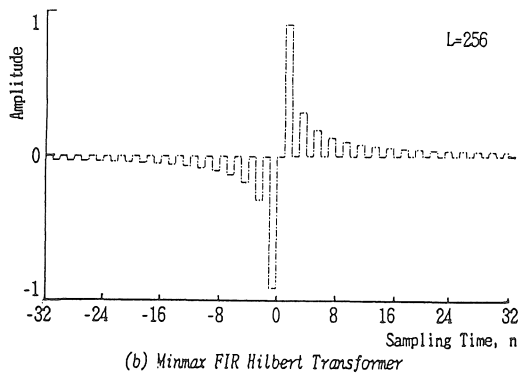
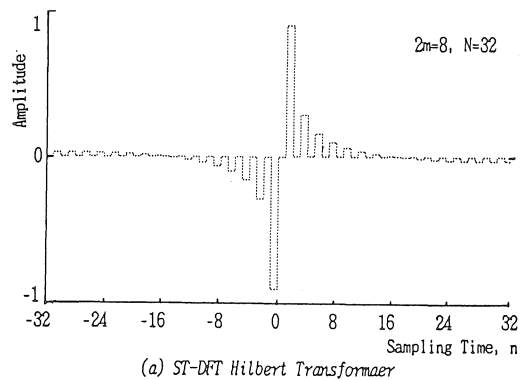


Fig.3 Comparison of unit sample response between the ST-DFT and minmax Hilbert transformers

where  $L^- = [n/R] - Q + 1,$   
 $L^+ = [n/R + Q],$  (15)

here,  $[A]$  represents the largest integer contained  $A,$   $\hat{\phi}_k(r)$  means the decimated instantaneous spectrum by every  $R$  sampling periods,

$\hat{\phi}_k(r) = \hat{\phi}_k(rR),$  (16)

and where  $f(n-rR)$  is such an interpolating filter as Lagrange, given by

$f(n-rR) = \frac{(-1)^{r+Q}}{(Q-1+r)!(Q-r)!(n/R-r)} \prod_{i=1}^Q (n/R+Q-i).$  (17)

As the summations are defined over finite terms both for  $k$  and  $r,$  eq.14 stands for interchanging the order of summations. Therefore,

$\hat{y}(n) = \sum_{r=L^-}^{L^+} f(n-rR) \{1/N \sum_{k=0}^{N-1} \hat{\phi}_k(r) W_N^{nk}\}$  (18)

So long as  $R \leq N,$  the Hilbert transformed output  $\hat{y}(n)$  are precisely regenerated from the decimated instantaneous spectrums. The summation for  $k$  on right hand of eq.18 represents IFT. The speeded up ST-DFT Hilbert transformed signals  $\hat{y}(n)$  are finally given by,

$\hat{y}(n) = 1/N \sum_{r=L^-}^{L^+} f(n-rR) \hat{y}_n(n),$  (19)

where  $\hat{y}_n(n) = \sum_{k=0}^{N-1} \hat{\phi}_k(rR) W_N^{nk}.$

6. RESULT OF COMPUTER SIMULATIONS

The ST-DFT Hilbert transformer is substantiated through computer simulations to be indicated in fig.4 by dotted curve as the parameter  $2m=8$  and  $N=32.$  The chained curve in the same figure shows that of pre-optimized minmax Hilbert transformer with the same 256 length to ST-DFT  $2m \times N.$  Both the simulations were carried by the CRAY X-MP at AIT to avoid roundoff error from simulations.

Figure 4 also shows that the ST-DFT transformer error does not exceed beyond that of pre-optimized minmax Hilbert transformer on the frequency domain of less than  $0.85\pi (= 3.4\text{kHz}$  of  $8\text{kHz}$  sampling) and guarantees that the phase shifting error is within  $5 \times 10^{-3}$  degree for the practical frequency domain of more than  $0.075\pi (= 0.3\text{kHz}$  of  $8\text{kHz}$  sampling) less than  $0.85\pi.$

As shown in fig.3(a), the unit sample response of ST-DFT transformer features of oscillation of its envelope being coincident with that of minmax transformer shown in fig.3(b). This oscillation of the envelope is induced with the  $\cos(\pi n/N)$  factor of eq.11. That is, the response of ST-DFT transformer oscillates on all the  $2mN$  sampling time with  $N$  period. On the other hand, minmax transformer response is non periodic on all the  $L$  sampling time. This periodicity of the ST-DFT transformer is seemed to improve the accuracy of phase shifting as shown in fig.4.

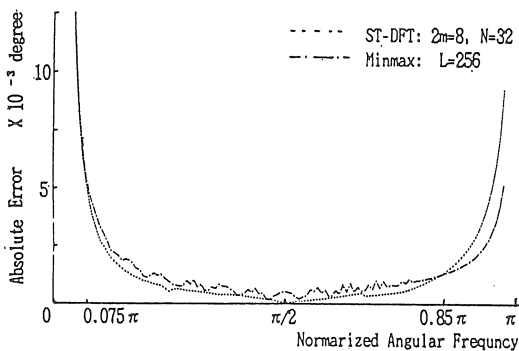


Fig.4 Phase shifting absolute error of the ST-DFT and minmax Hilbert transformers

**CONCLUSION**

A noble Hilbert transformer was discussed with emphasis on the instantaneous spectrum signal processing, through its circuitry configuration, speeding up processing, and phase shifting function. A primitive truncated Nyquist for the significant window  $h(*)$ , ST-DFT Hilbert transformer can exceed Rabiner's pre-optimized minmax one in both phase shifting accuracy and rapidness of transient response. Farther studies will improve such primitive instantaneous spectrum signal processing as done in minmax Hilbert transformer by Remez algorithm.

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