On Higher Differentials in Formal Power Series Rings

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形式的ベキ級数環における高階微分について

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Abstract. In [1], the concept of higher differentials in a commutative ring (by means of universal higher derivation) was introduced and it was shown that if a geometric regular local ring R is regular, then the submodule $A^n(R)$ of $A(R)$ generated by elements of degree n over R is R-free.

In this paper, we shall consider the case where R is a formal power series ring. When R is a residue class ring $k[[X_1, \cdots, X_s]]_{\mathfrak{q}}/\mathfrak{p}k[[X_1,\cdots, X_s]]_{\mathfrak{q}}$ where $\mathfrak{p}$, $\mathfrak{q}$ are prime ideals in $k[[X_1, \cdots, X_s]]$ such that $\mathfrak{p} \subset \mathfrak{q}$, we have the following result under some conditions: The submodule $A^n(R)$ of $A(R)$ generated by elements of degree n over R is R-free if R is regular.

1. Introduction. In the present paper, all rings are commutative rings with identity elements. A ring homomorphism will always mean a ring homomorphism which sends identity element to identity element.

Let R be a ring. By an R-module we understand an R-module in which 1_R, the identity element of R, operates as the identity operator.

A ring A will be called an R-algebra if R is an operator domain of A and there exist a ring homomorphism f from R into A such that the operation on A of an element $r \in R$ is given by the rule $r \cdot a = f(r)a$ for $a \in A$. f is called the structural homomorphism. Let P be a ring and let R be a P-algebra. A higher P-derivation from R into A is a family $\{d^n\}_{n \geqq 0}$ of P-linear mappings from R into an R-algebra A such that

(i) $d^0 a = a \cdot 1_A$ for every $a \in R$,

(ii) $d^n(ab) = \sum_{0 \leqq i \leqq n} d^i a \cdot d^{n-i} b$ for every $a, b \in R$ and $n \geqq 1$.

Let A be an R-algebra and let $\{d^n\}_{n \geqq 0}$ be a higher P-derivation from R into A. We call A (together with $\{d^n\}_{n \geqq 0}$) a higher differential algebra of R over P, when the following conditions are satisfied:

(1) As an R-algebra, A is generated by the elements $d^n a$ ($a \in R$, $n \geqq 0$) over R.

(2) For any higher P-derivation $\{\delta^n\}_{n \geqq 0}$ from R into an R-algebra N, there exists an R-algebra homomorphism φ from A into N which satisfies

$$\delta^n = \varphi d^n \quad \text{for } n \geqq 0.$$

In Proposition 1 of [1], it was already shown that, for any ring P and P-algebra R, there exists a higher differential algebra of R over P and it is uniquely determined up to an R-algebra isomorphism.

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From now on, we shall denote by $A_P(R)$ a higher differential algebra of R over P and by $\{d^n_{R,P}\}_{n\geq 0}$ the associated higher P-derivation from R into $A_P(R)$.

Denote by $A^n_P(R)$ the R-submodule of $A_P(R)$ generated by the elements

$$(d^{h_1}_{R,P}a_1)^{r_1} \cdots (d^{h_s}_{R,P}a_s)^{r_s}$$

over R where a_i's run through R and h_i's and r_i's are non negative integers such that $h_1r_1+\cdots+h_sr_s=n$ for some $s\geq 1$.

Denoting by Z the ring of rational integers, any ring R can be seen as a Z-algebra with the structural homomorphism $g: Z \longrightarrow R$ defined by $g(m)=m\cdot 1_R$ $(m\in Z)$. We shall write $A_Z(R)$ simply $A(R)$.

2. The higher differential algebra of a formal power series ring.

PROPOSITION. *Let $R=P[[X_\lambda]]_{\lambda\in\Lambda}$ be a formal power series ring in indeterminates X_λ $(\lambda\in\Lambda)$ over P. Then, introducing new indeterminates $X_{\lambda,n}$ $(\lambda\in\Lambda,\ n\geq 1)$, the higher differential algebra of R over P is given by the polynomial ring $A=R[[X_{\lambda,n}]]_{\lambda\in\Lambda,n\geq 1}$ over R and associated higher P-derivation $\{d^n\}$ is given by*

$$d^nX_\lambda=X_{\lambda,n} \text{ for } \lambda\in\Lambda \text{ and } n\geq 0$$

where $X_{\lambda,0}$ stands for X_λ.

PROOF. We can obtain this proof in almost the same way as Proposition 7 in [1]. Therefore we omit the proof.

COROLLARY. *If R is formal power series ring over P, then $A^n_P(R)$ is a free R-module for every $n\geq 0$.*

Let k be a field of characteristic p and denote by k_0 the prime field contained in k. Let B be the formal power series ring $k[[X_1,\cdots,X_s]]$ in s indeterminates over k, let $\mathfrak{p}\subset\mathfrak{q}$ be prime ideals of B and set $S=B_{\mathfrak{q}}$.
Let R be a residue class ring of S with respect to $\mathfrak{p}S$.

LEMMA 1. *With the notations as above, if R is regular and k is finitely generated over k_0, then $A^n(R)$ is R-free for every $n\geq 0$, Furthermore, for $p>0$. if R is regular and k finitely generated over a field k^q for some $q=p^m$ $(m>0)$ then $A^n(R)$ is R-free for every $n<q$.*

PROOF. Denoting by $\mathfrak{N}$ the maximal ideal $\mathfrak{q}S/\mathfrak{p}S$ of R, we shall show that

$$A^n(R)/\mathfrak{N}^rA^n(R)\cong A^n(R/\mathfrak{N}^s)\otimes_R(R/\mathfrak{N}^r)$$

for every $r\geq 1$ and $s\geq n+r$.
In fact, by Proposition 3 in [1],

$$A^n(R/\mathfrak{N}^s)=A^n(R)/V_n{}^{(s)}$$

where $V_n{}^{(s)}=\sum_{0\leq i\leq n} A^i(R)d^{n-i}\mathfrak{N}^s$ with $d^{n-i}=d^{n-i}_{R,Z}$.

It is clear that for $h\geq n$, $d^n\mathfrak{N}^h\subset\mathfrak{N}^{h-n}A^n(R)$ from the proof of Proposition 7 in [1]. Hence $V_n{}^{(s)}\subset\mathfrak{N}^rA^n(R)$ for every $s\geq n+r$.
Therefore we have

$$A^n(R/\mathfrak{N}^s) \underset{R}{\otimes} (R/\mathfrak{N}^r) \cong A^n(R/\mathfrak{N}^s)/\mathfrak{N}^r A^n(R/\mathfrak{N}^s)$$
$$\cong (A^n(R)/V_n^{(s)})/(\mathfrak{N}^r A^n(R)/V_n^{(s)})$$
$$\cong A^n(R)/\mathfrak{N}^r A^n(R).$$

Since R is a regular local ring containing k, the completion $\hat{R}$ of R contains a field $K \supset k_0$ ($K \cong R/\mathfrak{N}$) and $\hat{R} \cong K[[Y]]$ where $K[[Y]]$ is the ring of formal power series in $Y_1, \cdots, Y_l$ over K.

Therefore

$$\hat{R}/\hat{\mathfrak{N}}^r \cong R/\mathfrak{N}^r \cong K[[Y]]/(Y)^r \quad r \geqq 1.$$

Hence, for every $s \geqq n+r$, we have

$$A^n(R)/\mathfrak{N}^r A^n(R) \cong A^n(K[[Y]]/(Y)^s) \underset{K[[Y]]}{\otimes} (K[[Y]]/(Y)^r)$$
$$\cong A^n(K[[Y]])/(Y)^r A^n(K[[Y]])$$
$$\cong A^n(K[[Y]]) \underset{K[[Y]]}{\otimes} K[[Y]]/A^n(K[[Y]]) \underset{K[[Y]]}{\otimes} (Y)^r$$
$$\cong A^n(K[[Y]],) \underset{K[[Y]]}{\otimes} (K[[Y]]/(Y)^r).$$

By Proposition 5 in [1], we have

$$A^n(K[[Y]]) = A^n(K \underset{k_0}{\otimes} k_0[[Y]]) = \underset{0 \leqq i \leqq n}{\oplus} A^i(K) \underset{k_0}{\otimes} A^{n-i}(k_0[[Y]]).$$

Since, by the Corollary of Proposition, $A^n(K[[Y]])$ is clearly a free $K[[Y]]$-module for every $n \geqq 0$, $A^n(K[[Y]]) \underset{K[[Y]]}{\otimes} (K[[Y]]/(Y)^r)$ is a free $K[[Y]]/(Y)^r$-module for every $n \geqq 0$ and $r \geqq 1$. This implies that $A^n(R)/\mathfrak{N}^r A^n(R)$ is $R/\mathfrak{N}^r$-free for every $n \geqq 0$ and $r \geqq 1$. Since k is finitely generated over k_0 and R is finitely generated over k, $A^n(R)$ is a finite R-module for every $n \geqq 0$.

Hence, our assertion is obtained by Lemma 4 in [1].

If $p > 0$ and k is finitely generated over k^q, then by Proposition 4 and Proposition 10 in [1], we have

$$A^n(R) = A^n{}_{k^q}(R)$$

for $n < q$ and $A^n{}_{k^q}(R)$ is a finite R-module. Hence the assertion follows from Lemma 4 in [1].

Let $\mathfrak{p} = \mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_t = \mathfrak{q}$ be a maximal chain of prime ideals between $\mathfrak{p}$ and $\mathfrak{q}$ in B. Let k' be a field cotaining all the coefficients of formal power series $f_{i\nu_i}(X)$ $(i=0,1,\cdots t)$, where $\{f_{i\nu_i}(X)\}_{\nu_i \in \Lambda}$ is a base for $\mathfrak{p}_i$ and let $B' = k'[[X_1, \cdots, X_s]]$, $\mathfrak{p}_i' = \mathfrak{p}_i \cap B'$ $(i=0,1,\cdots,t)$, $S' = B'_{\mathfrak{q}'}$ and $R' = S'/\mathfrak{p}'S'$.

LEMMA 2. *Notations being as above, we assume that B is an integral extension of B'. If R is regular, then R' is regular.*

PROOF. First, we show that $\dim R = \dim R'$. Since B is integral over B', we have height $\mathfrak{q}$ = height $\mathfrak{q} \cap B'$ and height $\mathfrak{p}$ = height $\mathfrak{p} \cap B'$.

Then we get

$$\dim R = \dim S/\mathfrak{p}S = \text{height } \mathfrak{q} - \text{height } \mathfrak{p} = t$$
$$= \text{height } \mathfrak{q} \cap B' - \text{height } \mathfrak{p} \cap B' = \dim S'/\mathfrak{p}'S' = \dim R'.$$

Let $g_1, \cdots, g_t$ be maximal set of generators of maximal ideal $\mathfrak{N}$ of R.

Since $\mathfrak{q}$ is the ideal with a base consisting of formal power series in $k'[[X_1, \cdots, X_s]]$, we can assume that $g_1,\cdots,g_t\in\mathfrak{N}'$ and $(g_1,\cdots,g_t)R'=\mathfrak{N}'$ where $\mathfrak{N}'$ is a maximal ideal of R'. Thus R' is regular.

THEOREM. *Notations and assumptions being as in Lemma 2. We assume that k' is finitely generated over k^q in the case $ch(k)>0$ and over k_0 in the case $ch(k)=0$. If the local ring R is regular, then $A^n(R)$ is a free R-module for every $n\geqq 0$.*

PROOF. Now, we consider the $p>0$ and the case $p=0$ separately.

In the case $p>0$. Let Γ be a p-base of k and Δ a finite subset of Γ such that $k'=k^q(\Delta)$ contains all the coefficients of formal power series of a base for $\mathfrak{p}_i$ $(i=0,1,\cdots,t)$, where $q=p^m$ $(m\geqq 1)$. We put $k''=k^q(\Gamma-\Delta)$.

First we shall show that $k''\underset{k^q}{\otimes}R'\cong R$. Denoting by $(x_1,\cdots,x_s)$ the residue classes of $(X_1,\cdots,X_s)$ modulo $\mathfrak{p}'$, we have

$$Q(R') = k'((x_1,\cdots,x_s))$$

where $Q(R')$ and $k'((x_1,\cdots,x_s))$ are the quotient field of R' and $k'[[x_1,\cdots,x_s]]$ respectively. Since, by Lemma 2 in [1], Γ is q-independent over k^q, k' and k'' are linearly disjoint over k^q. By (22.3) in [3], $k=k'k''$ and $k'((x_1,\cdots,x_s))$ are linearly disjoint over k'. Hence k'' and $k'((x_1,\cdots,x_s))$ are linearly disjoint over k^q, thus we have

$$k''\underset{k^q}{\otimes}R'\cong k''R'\subset R$$

On the other hand, let $\bar{a}/\bar{b}$ $(\bar{a}\in B/\mathfrak{p},\ \bar{b}\in B/\mathfrak{p}-\mathfrak{q}/\mathfrak{p})$ be an element of R. Then we can write

$$\bar{a}=\Sigma\,\alpha_i\bar{a}_i, \qquad \bar{b}=\Sigma\,\beta_j\bar{b}_j$$

with $\bar{a}_i\in B'/\mathfrak{p}'$, $\bar{b}_j\notin\mathfrak{q}'/\mathfrak{p}'$ and $\alpha_i,\ \beta_j\in k''$. Since $\bar{b}^q=\Sigma\beta_j{}^q\bar{b}_j{}^q\in B'/\mathfrak{p}'-\mathfrak{q}'/\mathfrak{p}'$, we have

$$\bar{a}/\bar{b}=\bar{a}\,\bar{b}^{q-1}/\bar{b}^q\in k''(B'/\mathfrak{p}')_{\mathfrak{q}'/\mathfrak{p}'}=k''R'.$$

Hence $k''\underset{k^q}{\otimes}R'\cong R$. Next, we shall show $A^n(R)$ is R-free for $n<q$.
In fact, by Proposition 5 in [1], we have

$$A^n(R)=A^n{}_{k^q}(R)=\underset{0\leqq i\leqq n}{\oplus}\{A^i{}_{k^q}(k'')\underset{k^q}{\otimes}A^{n-i}{}_{k^q}(R')\}$$

in which each $A^i{}_{k^q}(k'')=A^i(k'')$ is k''-free and each $A^{n-i}{}_{k^q}(R')=A^{n-i}(R')$ is R'-free by Lemma 1. Hence each summand $A^i(k'')\otimes A^{n-i}(R')$ is R-free.
Therefore $A^n(R)$ is R-free for $n<q$. Thus $A^n(R)$ is R-free for every $n\geqq 0$.

In the case $p=0$. Since R is regular, by Lemma 2, R' is regular.
Hence, by Lemma 1, $A^n(R')$ is R'-free for every $n\geqq 0$. Let Λ be the trancendency base of k over k' and let $\tilde{k}=k_0(\Lambda)$. Since k' and $\tilde{k}$ are linearly disjoint over k_0, $\tilde{k}$ and $k'((x_1,\cdots,x_s))$ are linearly disjoint over k_0 by an argument as in the case $p>0$. Hence we have

$$\tilde{R}\cong\tilde{k}\underset{k_0}{\otimes}R'$$

where $\tilde{R}=\tilde{k}\,R'$. Therefore, by Proposition 5 in [1], we have

$$A^n(\tilde{R})=\underset{0\leqq i\leqq n}{\oplus}\{A^i(\tilde{k})\underset{k_0}{\otimes}A^{n-i}(R')\}$$

proving that $A^n(\tilde{R})$ is $\tilde{R}$-free for every $n\geqq 0$. We shall now show that

$$A(R)=R\underset{\tilde{R}}{\otimes}A(\tilde{R})$$

which implies that $A^n(R)$ is R-free for $n \geqq 0$. Let us put $R^* = \tilde{R}[k]$.

First, if k is finite algebraic over $k'' = k'\tilde{k}$, then $k = k''(\alpha)$ for some $\alpha \in k$ and we have $R^* = \tilde{R}[\alpha]$. Hence, by Lemma 1 in [1], we have

$$A(R^*) = R^* \underset{\tilde{R}}{\otimes} A(\tilde{R}).$$

Next, if k is not finite over k'', there exists a family $\{k_\lambda\}$ of finite algebraic extensions over k'' and $k = \cup k_\lambda$. As is shown above, $A(R_\lambda^*) = R_\lambda^* \underset{\tilde{R}}{\otimes} A(\tilde{R})$ and the associated higher k_0-derivation $\{d_\lambda{}^n\}_{n \geqq 0}$ from R_λ^* into $R_\lambda^* \underset{\tilde{R}}{\otimes} A(\tilde{R})$ is uniquely determined by $\{d^n_{\tilde{R}, k_0}\}_{n \geqq 0}$ and the following diagram is commutative,

$$\begin{array}{ccccc}
\tilde{R} & \xrightarrow{g^*_\lambda} & R^*_\lambda & \xrightarrow{h^*_{\mu\lambda}} & R^*_\mu \\
\downarrow d^n_{\tilde{R}, k_0} & & \downarrow d^n_\lambda & & \downarrow d^n_\mu \\
A(\tilde{R}) & \xrightarrow{u_\mu} & R_\lambda^* \underset{\tilde{R}}{\otimes} A(\tilde{R}) & \xrightarrow{h^*_{\mu\lambda} \otimes 1} & R_\mu^* \underset{\tilde{R}}{\otimes} A(\tilde{R})
\end{array}$$

where $u_\lambda(\omega) = 1 \otimes \omega$ and g^* is the canonical injection $\tilde{R} \longrightarrow R_\lambda^*$.
Accordingly we have a direct system $\{R_\lambda^* \underset{\tilde{R}}{\otimes} A(\tilde{R}),\ h^*_{\mu\lambda} \otimes 1\}$ and

$$A(R^*) = \underset{\longrightarrow}{\lim}\, R_\lambda^* \underset{\tilde{R}}{\otimes} A(\tilde{R}) \cong R^* \underset{\tilde{R}}{\otimes} A(\tilde{R}).$$

Since R^* is a subring of R containing B. R is a quotient ring of R^*.
Hence, by Proposition 8 in [1], we have

$$A(R) = R \underset{R^*}{\otimes} A(R_\lambda^*) \cong R \underset{\tilde{R}}{\otimes} A(\tilde{R}).$$

This completes the proof.

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