ON THE THEORY OF WASHING AND EXTRACTION

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When precipitates are washed with a definite quantity of water or some other solvent, it's not recommended to use the washings at the same time. The principle also applies to the extraction between two liquid phases. In almost all analytical textbooks this principle is introduced with the next formula:

$$C_n=\frac{C_0}{\left(1+\frac{1}{n\,a}\right)^n}$$

In some of them such special numbers, for instance, as $n=4$ and 5 are put in the formula to show $C_5<C_4$ practically. But the authors thought this is not the best way for the advanced students and tried to give a proof of $C_{n+1}<C_n$.

1. WASHING

If the total quantity of washings is divided in equal n parts and each of them is used for washing one after the other successively, the concentration of solution which remains absorbed by precipitates after the nth washing is to be expressed as follows:

$$C_n=\frac{C_0}{\left(1+\frac{1}{n\,a}\right)^n}\ ,\quad a=\frac{v}{V}$$

C_0 : original concentration of solution which remains with precipitates.

v : constant volume absorbed by precipitates.

V : total volume of washings.

2. EXTRACTION

Like the process in washing described in 1, the total extraction medium is divided in equal n parts and each of them is used for extraction one after the other successively. Then the present amount of solute which remains not extracted after the nth extraction, is to be expressed with the following formula:

$$W_n=\frac{W_0}{\left(1+\frac{1}{n\,b}\right)^n}\ ,\quad b=\frac{v\,d}{V},\quad d=\frac{W_1V}{nv(W_0-W_1)}$$

d : distribution constant.

v : constant volume of solution.

V : total quantity of extraction medium.

W_0 : original amount of solute.

3. PROOF OF $\left(1+\frac{1}{n+1}\cdot\frac{1}{a}\right)^{n+1}>\left(1+\frac{1}{n\,a}\right)^n$

C_n and W_n belong to the same types of formula as observed in 1 and 2. And the authors attempted te give the proof of $C_{n+1}<C_n$ by two different mathematical methods, using the binomial theorem and differential calculus.

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3.1 PROOF BY BINOMIAL THEOREM

$$\left(1+\frac{1}{n\ a}\right)^n=\sum_{k=0}^{n} {}_nC_k\left(\frac{1}{n\ a}\right)^k$$

$$\left(1+\frac{1}{n+1}\cdot\frac{1}{a}\right)^{n+1}=\sum_{k=0}^{n+1} {}_{n+1}C_k\left(\frac{1}{n+1}\cdot\frac{1}{a}\right)^k$$

$$=\sum_{k=0}^{n} {}_{n+1}C_k\left(\frac{1}{n+1}\cdot\frac{1}{a}\right)^k+\left(\frac{1}{n+1}\cdot\frac{1}{a}\right)^{n+1}$$

$${}_nC_k\left(\frac{1}{n\ a}\right)^k=\frac{n-1}{n}\cdots\cdots\frac{n-k+1}{n}\cdot\frac{1}{a^k k!}$$

$${}_{n+1}C_k\left(\frac{1}{n+1}\cdot\frac{1}{a}\right)^k=\frac{n}{n+1}\cdots\cdots\frac{n-k+2}{n+1}\cdot\frac{1}{a^k k!}$$

$$\frac{n-k+1}{n}\cdot\frac{n+1}{n-k+2}=1-\frac{k-1}{n^2-(k-2)n},\qquad \frac{k-1}{n^2-(k-2)n}>0\qquad (k>1)$$

$$\therefore\ 0<\frac{k-1}{n^2-(k-2)n}<1$$

$$\sum_{k=0}^{n+1} {}_{n+1}C_k\left(\frac{1}{n+1}\cdot\frac{1}{a}\right)^k>\sum_{k=0}^{n} {}_nC_k\left(\frac{1}{n\ a}\right)^k$$

3.2 PROOF BY DIFFERENTIAL CALCULUS

Put $t=\frac{1}{n\ a}$ in $\left(1+\frac{1}{n\ a}\right)^n$, then $f(t)=(1+t)^{\frac{1}{at}}$ and $\log f(t)=\frac{1}{a\ t}\log(1+t)$.

Put $g(t)$ in place of a $\log f(t)$, then $g(t)=\frac{\log(1+t)}{t}$ and $g'(t)=\frac{t-(1+t)\log(1+t)}{t^2(1+t)}$.

From $h(t)=t-(1+t)\log(1+t)$, $h'(t)=-\log(1+t)$, $0<t<\infty$ and $h(0)=0$ $\therefore$ $h'(t)<0$, $h(t)<0$

$\therefore$ $g'(t)<0$. From the result obtained above $g(t)$ is a decreasing function in the range $0<t<\infty$.

Then each of $\log f(t)$, $f(t)$ and $\left(1+\frac{1}{n\ a}\right)^n$ is also a decreasing function.

$\therefore$ $C_n=\dfrac{C_0}{\left(1+\frac{1}{n\ a}\right)^n}$ was proved to be an increasing function.

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