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On the Method of Approximate Solutions of Equations Combined the Bisection Method with the *Regula-Falsi* Method

二分法と割線法の併用による方程式の近似解法について

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Abstract. As the method of finding the approximate solution of the equation $f(x)=0$, the bisection method and the *regula-falsi* method are well known.

Making use of the above two methods simultaneously, we shall introduce new methods of the approximate solution prepared with the certainty of convergence of the bisection method and the speed of convergence of the *regula-falsi* method at the same time

1. Introduction.

Let $f(x)$ be continuous on the closed interval $[a,b]$. In this study we should like to introduce a new methods of giving the approximate solution of the equation $f(x)=0$, applicable to computer calculation.

When $f(x)$ is polynomial, we can find the true solutions easily by the factorization of $f(x)$. In the case that the function $f(x)$ can't be factorized, and that $f(x)$ be trigonometric, exponential or logarithmic function, the finding solution is much more difficult. So we must use the computer to search for the approximate solutions.

As the approximate calculation, ***the bisection method***, ***the secant method*** and ***Newton's method*** are well known.

With respect to the bisection method, the convergence of approximate sequence is proved using the convergent theorem of bounded and monotone sequence. But the speed of convergence is slow in general.

It is well known that the convergence of the secant method is faster in many cases than that of the bisection method. On the other hand, the assurance of the convergence of the secant method can't be proved in general.

As the Newton's method, the proof of convergence is done by using, for example, the principle of contraction mapping under the additional conditions on the smoothness of $f(x)$.

In our study, we try to use both methods of bisection and of the *regula-falsi* simultaneously. We introduce a methods of giving the approximate solutions prepared with the certainty of the convergence of the bisection method and the rate of convergence of the regula-falsi method at the same time.

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2. Preliminaries.

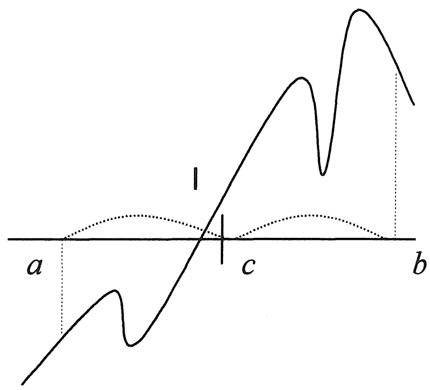
Let $f(x)$ be continuous on $[a,b]$ satisfying $f(a)\cdot f(b)<0$. By the mean value theorem, the exits at least one point $\alpha\in[a,b]$ such that $f(\alpha)=0$. But the concrete value of α is unknown. So we need to find the approximate solution of $f(x)=0$ instead of true solution α.

2.1 Bisection method.

Algorithm of the bisection method.

1. We find the interval $[a,b]$ such that $f(a)\cdot f(b)<0$.
2. Put $c=\dfrac{a+b}{2}$.
3. (I) When $f(c)\cdot f(a)>0$,

$$\begin{cases} a=c \\ b=b \end{cases}.$$



(ii) When $f(c)\cdot f(a)<0$

$$\begin{cases} a=a \\ b=c \end{cases}.$$

(iii) When $f(c)=0$, c is the desired solution of $f(x)=0$.

Repeating the procedures 1~3, we have sequences $[a_n,b_n]$ of intervals and $\{c_n\}$ of approximate solutions satisfying.

$$\text{(2-1)}\quad \begin{cases} a_n \leqq c_n \leqq b_n \\ b_n-a_n=\dfrac{1}{2^n}(b-a) \end{cases}.$$

Then $c=\lim c_n$ coincides with the true solution of $f(x)=0$.

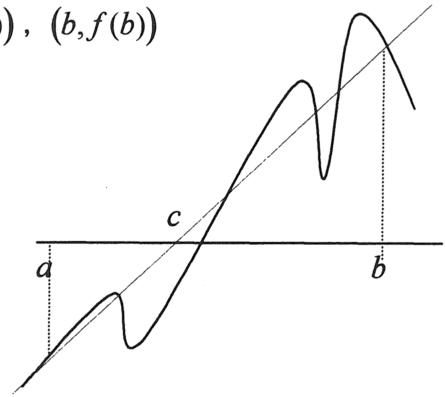
Remark. The sequences $\{a_n\},\{b_{n\}}\}$ are bounded and monotone. So, by the convergence theorem of Weierstrass, we can prove that $\{a_n\}$ and $\{b_n\}$ are convergent. Put $A=\lim a_n$ and $B=\lim b_n$. Then we have $A=B$ by (2-1). By virtue of the inequalities $a_n\le c_c\le b_n$, we can see that the sequence $\{c_n\}$ also convergent and that $C=A=B$, where C denotes the limit of $\{c_n\}$.

We emphasize here that the approximate sequence of $f(x)=0$ obtained by the bisection method is convergent for every continuous function $f(x)$.

2-2 Method of regula-falsi.

Algorithm of the method of regula-faisi.

1. We find the interval $[a,b]$ such that $f(a)\cdot f(b)<0$.
2. The equation of straight line joining 2 points $(a,f(a))$, $(b,f(b))$ is as follows.



$$y=f(a)+\frac{(f(b)-f(a))(x-a)}{b-a}.$$

Let c be the intersection point of the above Line and X-axis. Then

$$c=\frac{af(b)-bf(a)}{f(b)-f(a)}=a-\frac{f(a)}{f(b)-f(a)}(b-a).$$

3. (I) When $f(c)\cdot f(a)>0$

$$\begin{cases} a=c \\ b=b \end{cases}.$$

(ii) When $f(c)\cdot f(a)<0$

$$\begin{cases} a=a \\ b=c \end{cases}.$$

(iii) When $f(c)=0$, c is the desired solution of $f(x)=0$.

Repeating the procedures 1~3, we have sequences $[a_n, b_{n]}$ of intervals and $\{c_n\}$ of approximate solutions satisfying

(2-2) $$c_n=\frac{a_n f(b_n)-b_n f(a_n)}{f(b_n)-f(a_n)},$$

(2-3) $$c_n=a_n-\frac{f(a_n)}{f(b_n)-f(a_n)}(b_n-a_n),$$

(2-4) $$c_n=b_n-\frac{f(b_n)}{f(b_n)-f(a_n)}(b_n-a_n),$$

(2-5) $$a_n<c_n<b_n.$$

Under some additional conditions on the smoothness of $f(x)$, we can prove that $\{c_n\}$ converges to the true solution of $f(x)=0$. But we can't prove the convergence of approximate sequence for general continuou function.

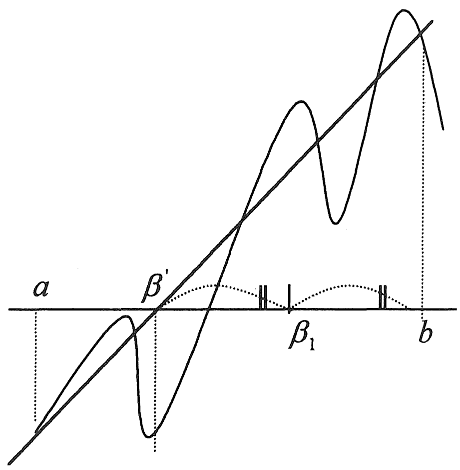
3. Method (1) of approximate solution combined the bisection method with the method of regula − falsi.

Algorithm of the combined method (1).

We consider only the case that $f(a) < 0 < f(b)$.

Step 1.

1. Put $\beta_1' = a - \dfrac{f(a)}{f(b) - f(a)}(b-a)$.



2. (i) When $f(\beta_1') > 0$, put

$$\begin{cases} a_1' = a \\ b_1' = \beta_1' \end{cases}.$$

(ii) When $f(\beta_1') < 0$, put

$$\begin{cases} a_1' = \beta_1' \\ b_1' = b \end{cases}.$$

(iii) When $f(\beta_1') = 0$, β' is the desired solution of $f(x) = 0$.

3. Put $\beta_1 = \dfrac{a_1' + b_1'}{2}$ (first approximate solution).

(i) When $f(\beta_1) > 0$, put

$$\begin{cases} a_1 = a_1' \\ b_1 = \beta_1 \end{cases}.$$

(ii) When $f(\beta_1) < 0$, put

$$\begin{cases} a_1 = \beta_1 \\ b_1 = b_1' \end{cases}.$$

(iii) When $f(\beta_1) = 0$, β_1 is the desired solution of $f(x) = 0$ and we have.

$$\begin{cases} a_1 \leqq \beta_1 \leqq b_1 \\ b_1 - a_1 = \dfrac{1}{2}(b_1' - a_1') < \dfrac{1}{2}(b-a) \end{cases}.$$

Step 2.

1. Put $\beta_2' = a_1 - \dfrac{f(a_1)}{f(b_1)-f(a_1)}(b_1 - a_1)$.

2. (i) When $f(\beta_2') > 0$, put

$$\begin{cases} a_2' = a_1 \\ b_2' = \beta_2' \end{cases}.$$

(ii) When $f(\beta_2') < 0$, put

$$\begin{cases} a_2' = \beta_2' \\ b_2' = b_1 \end{cases}.$$

(iii) When $f(\beta_2') = 0$, β_2' is the desired solution of $f(x) = 0$.

3. Put $\beta_2 = \dfrac{a_2' + b_2'}{2}$ (the second approximate solution)

(i) When $f(\beta_2) > 0$, put

$$\begin{cases} a_2 = a_2' \\ b_2 = \beta_2 \end{cases}.$$

(ii) When $f(\beta_2) < 0$, put

$$\begin{cases} a_2 = \beta_2 \\ b_2 = b_2' \end{cases}.$$

(iii) When $f(\beta_2) = 0$, β_2 is the desired solution of $f(x) = 0$ satisfying

$$\begin{cases} a_2 \leqq \beta_2 \leqq b_2 \\ b_2 - a_2 = \dfrac{1}{2}(b_2' - a_2') < \dfrac{1}{2^2}(b-a) \end{cases}$$

Step n.

Repeating the procedures, we have the n-the approximate solution β_n satisfying

$$(3-1) \qquad \begin{cases} a_n \leqq \beta_n \leqq b_n \\ b_n - a_n = \dfrac{1}{2}(b_n' - a_n') < \dfrac{1}{2^n}(b-a) \end{cases}$$

4. Convergence of the approximate sequence obtained by the combined method (1).

In the argument of classical secant method or the method of *regula-falsi*, thel imit of approximate sequence $\{\alpha_n\}$ does not necessarily exist. But by our method (1), the sequence $\{\beta_n\}$ of approximate solutions is always convergent to the true solution of $f(x)=0$. Indeed we have the following

Theorem 1. *Let* $f(x)$ *be a continuous function on a closed interval* $[a,b]$. *Suppose that* $f(a)\cdot f(b)<0$, *then the sequence* $\{\beta_n\}$ *of approximate solutions obtained by the combined method (1), converges always to the true solution of the equation* $f(x)=0$.

Proof. When $f(a)<0<f(b)$, we have

$$a \leqq a_1 \leqq a_2 \leqq \cdots \leqq a_n \leqq \cdots \leqq b_n \leqq \cdots \leqq b_2 \leqq b_1 \leqq b$$

Then $\{a_n\}$ is bounded above and monotone increasing and therefore, by virtue of **Weierstrass theorem**, $A=\lim a_n$ exists. Similarly, $B=\lim b_n$ also exists. Then

$$0 \leqq B-A=\lim(b_n-a_n)\leqq \lim\frac{1}{2^n}(b-a)=0 \quad \text{and therefore} \quad A=B.$$

By virtue of the continuity of $f(x)$, we have

$$f(A)=\lim f(a_n)\leqq 0$$

$$f(B)=\lim f(b_n)\geq 0$$

and hence $f(A)=f(B)=0$.

By the inequalities $a_n \leq \beta_n \leq b_n$ in (3−1), we have $A=\lim a_n \leqq \lim\beta_n \leqq \lim b_n = B = A$.

Therefore $\beta=\lim\beta_n$ exist and the equalities $\beta=A=B$ and $f(\beta)=f(A)=0$ hold.

Consequently $\{\beta_n\}$ converges to the true solution β of $f(x)=0$.

By the same manner, we can also prove our theorem in the case that $f(a)>0>f(b)$.

Thus the proof of our theorem is completed. ■

Remark. The convergence of approximate sequence by the method of regula-falsi isn't be proved in general. But, adopting our method (1), we can add to the method of regula-falsi the assurance of the convergence of approximate sequence of the equation $f(x)=0$ for every continuous function $f(x)$.

5. Method (2) of approximate solution combined the bisection method with the method of regula-falsi.

Algorithm of the combined method (2).

It suffices to consider the case that $f(a) < 0 < f(b)$.

Step 1.

1. Put $c = \dfrac{a+b}{2}$.
2. (i) When $f(c) \cdot f(a) < 0$, put $a_1 = c, \quad b_1 = b$.

   (ii) When $f(c) \cdot f(a) > 0$, put $a_1 = a, \quad b_1 = c$.

   (iii) When $f(c) = 0$, c is the desired solution of

3. We denote by γ_1 the approximate solution satisfying (2-2)~(2-5) for a_1 and b_1 satifying

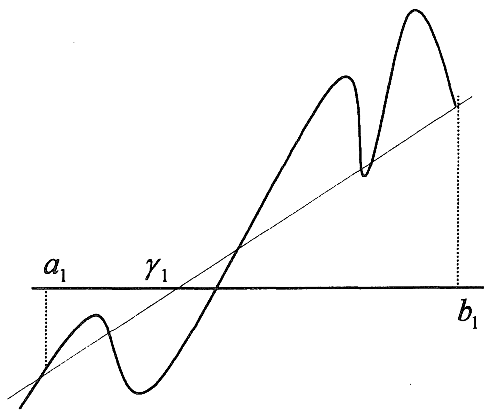
(5-1) $$\gamma_1 = \frac{a_a f(b_1) - b_1 f(a_1)}{f(b_1) - f(a_1)}$$

(5-2) $$\gamma_1 = a_1 - \frac{f(a_1)}{f(b_1) - f(a_1)}(b_1 - a_1)$$

(5-3) $$\gamma_1 = b_1 - \frac{f(b_1)}{f(b_1) - f(a_1)}(b_1 - a_1).$$

Then we have

$$\begin{cases} a_1 < \gamma_1 < b_1 \\ f(a_1) < 0 < f(b_1) \\ b_1 - a_1 = \dfrac{1}{2}(b-a) \end{cases}$$



Step 2.

1. (i) When $f(\gamma_1) > 0$, put $a_2' = a_1$ and $b_2' = \gamma_1$.

   (ii) When $f(\gamma_1) < 0$, put $a_2' = \gamma_1$ and $b_2' = b_1$.

   (iii) When $f(\gamma_1) = 0$, γ_1 is the desired solution of $f(x) = 0$

   (iv)

2. Put $c_2' = \dfrac{a_2' + b_2'}{2}$.

   (i) When $f(c_2') > 0$, put $a_2 = a_2'$ and $b_2 = c_2'$.

   (ii) When $f(c_2') < 0$, put $a_2 = c_2'$ and $b_2 = b_2'$.

   (iii) When $f(c_2') = 0$, c_2' is the desired solution of $f(x) = 0$.

3. We denote by γ_2 the third approximate solution satisfying for (5-1)~(5-3) for a_2 and b_2.
   Then we have

$$\begin{cases} a_2 < \gamma_2 < b_2 \\ f(a_2) < 0 < f(b_2) \\ b_2 - a_2 < \dfrac{1}{2^2}(b-a) \end{cases} .$$

....

Step n.

Repeating the procedures, we obtain the n-th approximate solution γ_n and the interval $[a_n, b_n]$ satisfying

$$\text{(5-4)} \quad \begin{cases} a_n < \gamma_n < b_n \\ f(a_n) < 0 < f(b_n) \\ b_n - a_n < \dfrac{1}{2^n}(b-a) \end{cases} .$$

6. Convergence of approximate sequence obtained by the combined method (2)

In this section, we should like to show that the approximate sequence by the method (2) converges for every continuous function.

Theorem 2. *Let $f(x)$ be continuous on [a, b]. Suppose that $f(a)\cdot f(b)<0$, then $\{\gamma_n\}$, the sequence of approximate solutions obtained by the method (2) converges to the true solution of $f(x)=0$ for every continuous function $f(x)$.*

Proof. Suppose that $f(a)<0<f(b)$. We denote by $\{[a_n,b_n]\}$ the intervals satisfying the inequalities

$$(5\text{-}4)\qquad \begin{cases} a_n<\gamma_n<b_n \\ f(a_n)<0<f(b_n) \\ b_n-a_n<\dfrac{1}{2^n}(b-a) \end{cases}$$

and the inequalities

$$a\leqq a_1\leqq a_2\leqq\cdots\leqq a_n\leqq\cdots\leqq b_n\leqq\cdots\leqq b_2\leqq b_1\leqq b.$$

Then $\{a_n\}$ is bounded above and monotone increasing and therefore, by virtue of the convergence theorem of Weierstrass, $A=\lim a_n$ exists. Similarly, $B=\lim b_n$ also exists. Then

$$0\leqq B-A=\lim(b_n-a_n)\le\lim\frac{1}{2^n}(b-a)=0$$

This implies that $A=B$. By the continuity of $f(x)$, we have

$$f(A)=\lim f(a_n)\leqq 0$$
$$f(B)=\lim f(b_n)\geqq 0$$

and hence $f(A)=f(B)=0$.

By the inequalities $a_n<\gamma_n<b_n$, we have also $A=\lim a_n\leqq\lim\gamma_n\leqq\lim b_n=B$. Therefore, $\gamma=\lim\gamma_n$ exist and is equal to both A and B. And further we obtain

$$\gamma=\lim\gamma_n=A=B,$$
$$f(\gamma)=f(A)=f(B)=0$$

Thus $\gamma=\lim\gamma_n$ is the desired true solution of $f(x)=0$.

By the same way, we can also prove our theorem in the case that $f(a)>0>f(b)$.

This completes the proof. ∎

7. The acceleration of convergence of approximate sequence obtained by the method (2).

Finally we shall show that the speed of convergence of our method (2), is faster than that of the bisection method.

Theorem 3. Let $f(x)$ be in the class $c^2[a,b]$. Suppose that $f(a)\cdot f(b)<0$ and $f'(x)\neq 0$ In (a,b). Then the speed of the convergence of the sequence $\{\gamma_n\}$ obtained by our method (2) is faster than that of the bisection method..

Proof. The n-th approximate solution γ_n satisfies the following equalities

$$\begin{aligned}\gamma_n &= a_n - \frac{f(a_n)}{f(b_n)-f(a_n)}(b_n-a_n)\\ &= a_n - \frac{f(a_n)}{f'(c_n)(b_n-a_n)}(b_n-a_n) \quad (\because \text{mean value theorem})\\ &= a_n - \frac{f(a_n)}{f'(c_n)}\\ &= a_n - \frac{f(a_n)-f(\alpha)}{f'(c_n)} \qquad (\because f(\alpha)=0).\end{aligned}$$

Then

$$\begin{aligned}\gamma_n-\alpha &= a_n-\alpha-\frac{f(a_n)-f(\alpha)}{f'(c_n)}\\ &= a_n-\alpha-\frac{f'(d_n)(a_n-\alpha)}{f'(c_n)} \qquad \text{(mean value theorem)}\\ &= (a_n-\alpha)\left\{1-\frac{f'(d_n)}{f'(c_n)}\right\}\\ &= \frac{a_n-\alpha}{f'(c_n)}\{f'(c_n)-f'(d_n)\}\\ &= \frac{a_n-\alpha}{f'(c_n)}f''(\xi_n)(c_n-d_n)\end{aligned}$$

Hence

$$|\gamma_n-\alpha| = \left|\frac{f''(\xi_n)}{f'(c_n)}\right||a_n-\alpha||c_n-d_n| \leqq |M||a_n-\alpha|\frac{1}{2^n}|b-a|.$$

The quantity $\frac{1}{2^n}|b-a|$ is the term concerning with the speed of the bisection method. By theorem2, we have $\lim|a_n-\alpha|=0$, consequently the above inequality shows that the speed of our method (2) is faster than that of the bisection method. ∎

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