On the Steady Magnetic Field due to the Retation of a Polarized Dielectric Cylinder

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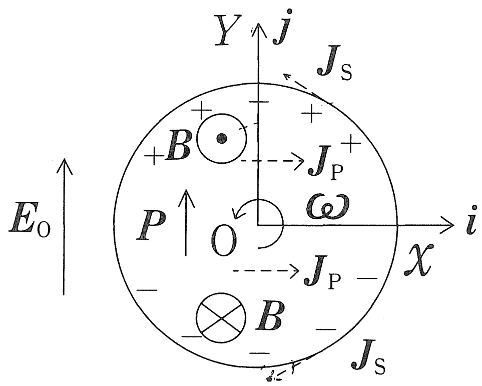
分極した誘電体円柱の回転による定常磁界について

荒　川　泰　二

According to the electromagnetic formulations of moving media, we discuss the steady magnetic field and current distribution produced when a long dielectric cylinder is spun about its axis in a uniform electric field applied perpendicular to the axis.

As one of the supplements to the paper[1] on the concept of "hidden momentum" introduced by W. Shockley and H. P. James,[2] we discuss here on the steady magnetic field produced when a long dielectric cylinder (scalar permittibity ϵ, permeability $\mu \approx \mu_0$ and electric conductivity $\sigma = 0$) is spun with constant angular velocity ω about its axis (z axis) in a uniform electric field $\boldsymbol{E}_0$ applied perpendicular to the axis.



There are several formulations of elecrtodynamics of moving media, compatible in spite of differences in forms.[3] Here, the Minkowski's theory is mainly used, which was the first formulation for moving media and is still best known. And we neglect the end effect of the long cylinder, the inertia of the matter and the change of its macroscopic property by rotation.

In the laboratory frame, according to the Minkowski formulation, we have the following constitutive relation when $\epsilon_0\mu_0 v^2$ is omitted and μ is assumed to be μ_0;

$$\boldsymbol{B} = \mu_0\{\boldsymbol{H} - (\epsilon - \epsilon_0)(\boldsymbol{v} \times \boldsymbol{E})\}, \tag{1}$$

where $\boldsymbol{B}, \boldsymbol{H}$ and $\boldsymbol{E}$ are the magnetic flux density, the magnetic field intensity and the electric field intensity at the point with a velocity $\boldsymbol{v}$ in the cylinder.

From Eq. (1), together with $\mathrm{div}\boldsymbol{B} = 0$, $\mathrm{rot}\boldsymbol{E} = 0$ (because of the steady field) and $\mathrm{rot}\boldsymbol{v} = 2\boldsymbol{\omega}$, it follows[4] that

$$\begin{aligned} \mathrm{div}\boldsymbol{H} &= (\epsilon - \epsilon_0)(\boldsymbol{E} \cdot \mathrm{rot}\boldsymbol{v} - \boldsymbol{v} \cdot \mathrm{rot}\boldsymbol{E}) \\ &= 2(\epsilon - \epsilon_0)\boldsymbol{E} \cdot \boldsymbol{\omega} . \end{aligned} \tag{2}$$

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We assume that $\boldsymbol{E}$ is unchanged by the rotation, namely, equal to

$$\boldsymbol{E} = 2\epsilon_0 \boldsymbol{E}_0/(\epsilon + \epsilon_0). \tag{3}$$

Then the direction of $\boldsymbol{E}$ is perpendicular to $\boldsymbol{\omega}$. From Eq. (2)

$$\mathrm{div}\boldsymbol{H} = 0.$$

Therefore, noting that rot$\boldsymbol{H}$ is zero in the steady field with no convection and conduction currents, it is reasonable to regard the possible solution of $\boldsymbol{H}$ as zero in the present case. Thus, inside the cylinder

$$\boldsymbol{B} = \mu_0(\boldsymbol{P} \times \boldsymbol{v}), \tag{4}$$

where the electric polarization $\boldsymbol{P}$ may be given by

$$\boldsymbol{P} = (\epsilon - \epsilon_0)\boldsymbol{E} = \frac{2\epsilon_0(\epsilon - \epsilon_0)}{\epsilon + \epsilon_0} \boldsymbol{E}_0. \tag{5}$$

Inserting $\boldsymbol{P} = P\boldsymbol{j}$ and $\boldsymbol{v} = \omega(-y\boldsymbol{i} + x\boldsymbol{j})$ to Eq. (4), we obtain

$$\boldsymbol{B} = \mu_0 P\omega y \boldsymbol{k}. \tag{6}$$

Here x and y are the rectangular coordinates of the point considered; $\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$ are unit vectors parallel to the rectangular axes.

(In the Chu formulation,[3] the magnetic field intensity $\boldsymbol{H}_c$ in this case is given by $\boldsymbol{P} \times \boldsymbol{v}$. This $\boldsymbol{H}_c$ is different from Minkowski's $\boldsymbol{H}$ obtained above. However, the observable quantity in the electromagnetic induction due to the transition of the cylinder at rest to its steady rotation is $\mu_0 \boldsymbol{H}_c$, which is equal to Minkowski's $\boldsymbol{B}$.)

In free space outside the cylinder, neglecting the end effect, the magnetic flux density $\boldsymbol{B}(= \mu_0 \boldsymbol{H})$ is zero because of the continuity of B_n (normal component of $\boldsymbol{B}$) and H_t (tangential component of $\boldsymbol{H}$) across the cylindrical interface.

Next, we will consider the steady current distribution in the cylinder. In the Minkowski formulation, the current density $\boldsymbol{J}$ $[= \rho \boldsymbol{v} + \sigma(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})]$ is zero in the case of true charge density $\rho = 0$ and $\sigma = 0$. On the other hand, in the Chu formulation regarding a polarized dielectric medium as containing a large number of small electric dipoles, the polarization current density $\boldsymbol{J}_p = \partial \boldsymbol{P}/\partial t + \mathrm{rot}(\boldsymbol{P} \times \boldsymbol{v})$ contributes to rot$\boldsymbol{H}_c$(= rot$\boldsymbol{B}/\mu_0$), together with $\epsilon_0 \partial \boldsymbol{E}/\partial t$ and $\boldsymbol{J}$. In the steady field now considered, we have

$$\boldsymbol{J}_p = \mathrm{rot}(\boldsymbol{P} \times \boldsymbol{v}) = P\omega \boldsymbol{i}. \tag{7}$$

Eq. (7) shows the existence of the uniform current parallel to the x-axis in the cylinder. Further, there is the surface current along the cylindrical surface whose density is

$$\boldsymbol{J}_s = P\omega y \boldsymbol{\varphi}_1, \tag{8}$$

where $\boldsymbol{\varphi}_1$ is the azimuthal unit vector of cylindrical coordinate. And it should be noted that the polarized charge distribution on the cylindrical surface is conserved by the interior current and surface current shown in Eq. (7) and (8).

References

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