Functional Approximations of Various Characteristic Curves of Electronic Elements and Generalization of Equivalent Linearization and Describing Function Method

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Summary

Until today, equivalent linearization and describing function method have been known as a very powerfull tool for analyzing the nonlinear systems. However the application was restricted to only 2nd or 3rd ordersystem. In this paper, it was shown that Krylov-Bogoliubov method could be easily generalized to the nth order system using the polar-coordinates of n dimensional Euclidean R^n space, and equivalent linearization method was equivalent to take mean value in one functional space, For this analysis, the concepts of functional analysis, especially, the concept of L. Schwartz's distributions, Dirac's Dlta function $\delta(t)$, and the normed vector (functional) space etc., play important roles. Moreover, it will be shown that the knot theory on topology has some relation to the limit cycle of periodic solutions. This method and the concepts will be used for the nonlinear analysis in the near future.

1. Preparation

1. 1. Functional Space: D, D', E, E', C etc.

A functional space F is one of vector spaces, but being different from finite-dimensional vector space, its suffix is infinite, *i.e.* real number. It can be defined as follows :

1° if $f(x)\in F$, and $\lambda\in K$, then
$\lambda f(x)\in F$ where K is a field, which is practically real or complex numbers.

2° if $f_1(x)$, $f_2(x)\in F$ and λ, $\mu\in K$, then
$\lambda f_1+\mu f_2\in K$, also.

the following property can be readily verified from the the above definition:

3° if $(f_k)_{k\in I}\in F$, then $\sum_{k\in I}\lambda_k f_k\in F$ for any $(\lambda_k)_{k\in I}$. $\subset K$.

Generally, the functional space and its subspace, and the dual space are the convenient concepts for approximating the characteristic curves of electronic elements and for the equivalent linearization as will be described later. Next, we specify some particular such spaces :

1. C : a space consisting of every continuous functions.
2. C^m : space, which contains all functions, having the continuous derivatives to "the mth order".
3. D and its dual space D' : a space consists of all functions on R^n, which are infinite differentiable, and have bounded supports. Its dual space D' is continuous linear form on D ; *i.e.* $<T, \varphi>$, $T\in D$, $\varphi\in D'$. The true useful spaces are the subspaces of $D$and D', such as D_+, $D(\Gamma)$, etc·· Concerning these spaces, L. Schwartz has already explaines in his book [1] in detail.
4. E and its dual space E' : a space consists of all functions, which are only infinite differentialable, but not necessariy bounded supports.
5. $L^2[a, b]$: all functions, which are measurable and the square functions are integrable :

$$(1.1\,;\,1)\quad \int_a^b |f(x)|^2 dx<+\infty,$$

The reasons, why these functional spaces are very usefull are, that, the spaces are vector spaces, in general, the linear or nonlinear operator transforms the one functional space to the another functional space. Therefore, where both such functional spaces have been known, using a family of some points in "one" space (i.e. for example, $a_0\cos\omega t$, $\forall a_0$ in Hilbert space),

One can approximate "the operator" : i.e. which transform the points in one space to the co-respondig family of points in another space. This is the concept of equivalent linearization method. Furthermore, it is desired that these spaces have distance or norm. Next, we explain about "the norm"

1. 2. Norm

"Norm" is defined as follows :

(1.2; 1) $\|x\|_p=(\sum_{i\in I}|x_i|^p)^{1/P}$

or,

$$\|f(x)\|_{L^P}=(\int|f(x)|^P dx)^{1/P},\quad 1\leqq P\leqq\infty$$

The norm, whose p is one, is integrable or local integrable function space. And the norm, whose p is infinite equalsts the norm : $\sup_{x\in I}|f(x)|$. These two norms are not differentiable norms. The other norms : $1<p<\infty$ norms, are differentiable, especially, in the case of $p=2$, normed space is equivalent to Hilbert space: complex, completness, and inner product space. In Hilbert space, orthogonallity condition is satis fied : i.e., (1.2; 2) Hilbert space has the orthonormal basis : $\{ei\}_{i\in I}$, and,

$$<f_i,\ f_j>=0,\ \text{if } i\neq j.$$
$$=1,\ \text{if } i=j.$$

and, in the space, any vector can be expressed with the basis : i.e., if $f(x)\in L^2$, then $f(x)$ can be expressed as the following form :

(1.2; 3) $f(x)=\sum_{i\in I}\lambda_i f_i(x)$,

and this expansion is unique and exists in any case Therefore, until today, only this space has been used in approximation theory. But we suppose that other spaces, such as Banach space, will be useful in the nonlinear analysis in the future. And further, the norm, which contains the derivative until the mth order will be used as follow:

(1.2; 3) $\|f(x)\|_{L^P,m}\triangleq\left(\sum_{|S|\leq m}\int_V|D^S f(x)|^p_{dx}\right)^{1/p}$

where $D^S=d^S/dt^S$ or $=\partial^{S_1+S_2+\cdots+S_n}f(x)/\partial x_1{}^{S_1}\cdots\partial x_n{}^{S_n}$, and $|S|=\sum_{i=1}^n S_i$.

2. Functional Approximations

2. 1. Given Functions or Given Data

Usually, givin approximating functions are specified with the experimental data or from the physical law. For example, Esaki diode v-i curve can be physically represented as the following form :

(2.1; 1) $i_D/I_P=Ave^{-Bv}+De^{Fv}+Gv$,

where :

I_P is peak current,

V_p is peak voltage, $v/V_P\rightrightarrows v$,

A, B, D, F, and G are constants. nrespectively The term ve^{-v} means the tunneling current of the diode, and e^v is the injection current, which appear in the high-voltage positive-resistance branch, and further the term v means that the current equals to zero in the neighorhood of the valley current. But, in general, the following two cases are distinct mathematically, :

Case 1. (experimental data are given) : i.e.

$$(2.1;2)\begin{cases} D_0=\{(x_1{}^{(0)},\ y_1{}^{(0)}),\ (x_2{}^{(0)},\ y_2{}^{(0)}),\cdots, \\ \qquad (x_p{}^{(0)}_o,\ y_p{}^{(0)}_o) \\ D_1=\{(x_1{}^{(1)},\ y_1{}^{(1)},\cdots,\ (x^p{}_{(1)1},\ y^p{}_{(1)1})\} \\ \qquad\cdots\cdots\cdots \\ D_n=\{(x_1{}^{(n)},\ y_1{}^{(n)}),\cdots,\ (x^p{}_{(n)n},\ y^p{}_{(n)n})\} \end{cases}$$

where, $x_i{}^{(j)}$ means the ith coordinates on the jth data, and $y_i{}^{(j)}$ means that the jth derivatives of y function of the x variable on the i th Y coordinates. Fom these data, we must determine. the form of the function $y=y(x)$.

Case 2. In this case, from the physical meaning, the form of the function $y=y(x)$ is somewhat known. Andthen at first, we may specaify the form $y=y(x,A,B\cdots)$, and then the constant $A,B,\cdots$, etc. must be determined from the other data.

In both cases, we want to utilize the given informations as possible as. and there fore, once selecting one particular data, the (functional) space or norm are different from the other case. Hereafter, the (given) specified function is denoted as follows :

(2.1; 3) G (is specified from the given data)

2. 2. Specification of the Functional Approximations.

Once given the approximated function G, we define the distance between G and $F_{(i)}(x;\alpha,\beta,\cdots)$, and it is deonted as follows:

(2.2; 1) $d(F_{(i)}(x;\alpha,\beta,\cdots),\ G)$,

Generally, the distance d equal to the norm of the $d$ifference : $F_{(i)}-G$:

(2.2; 2) $d(F_{(i)},\ G)\triangleq\|F_{(i)}-G\|$,

Then, the problem may be specified as follows. i.e., the distance :

(2.2; 3) $\min_{\alpha,\beta,\cdot\cdot} d(F_{(i)}(x;\alpha,\beta\cdot\cdot), G)$

mininiz by determining the suitable value of each parameters $\alpha,\beta,\cdots$.

Find such parameters values. This problem is quite equivalent to the linear or genmetric programming one. In that case the following points must be specified : that is,

(Question 1) what function $F_{(i)}$ must be selec ted? (Question 2) what distance is selected?

Usually a suitable norm is specified in Banach (or Hilbert space).

(Question 3) The number of the parameters $\alpha,\beta,\cdot\cdot$. Or, in other words, how many numbers of parameters must be taken for the fairly better approximations?

2.3. Linear Functional Approximations.

Now it may be defined that linear approximations may be expressed as follows :

(2.3; 1) $F_{(i)}(x;\lambda\alpha,\lambda\beta,\cdot\cdot)=\lambda F_{(i)}(x;\alpha,\beta,\cdots)$, for $\forall\lambda$then,

the particular examples will be mentiond as follows :

example 1. the space : $\{e^{ikx};\ k=0,\pm1,\cdot\cdot\}$

(2.3; 2) $F_{(1)}(x)\triangleq\sum_{k=-\infty}^{+\infty} c_k e^{ikwx}$,

example 2. the space:

$\{\cos k\omega x,\ \sin k\omega x;\ k=0,1,\cdot\cdot\}$

(2.3; 3) $F_{(2)}(x)\triangleq a_0+\sum_{k=1}^{\infty}(a_k\cos k\omega x+b_k\sin k\omega x)$

the coefficients a_k,b_k and c_k are clearly related mutually as follows :

(2.3; 4) $a_0=c_0,\qquad c_0=a_0$

$k>0,\ \begin{cases} a_k=c_k+c_{-k} \\ b_k=i(c_k-c_{-k}) \end{cases} \qquad \begin{cases} c_k=\dfrac{a_k-ib_k}{2} \\ c_{-k}=\dfrac{a_k+ib_k}{2} \end{cases}$

example 3. the ortho normal series : $\{P_n(x);\ n=0,\ 1,\ \cdot\cdot\}$

(2.3; 5) $F_{(3)}\triangleq\sum_{k=0}^{\infty}\lambda_k P_k(x)$,

particullary, if $P_n(x)=x^n$,

(2.3; 6) $F_{(4)}\triangleq\sum_{k=0}^{\infty}\lambda_k x^k$, or $\sum_{k=-\infty}^{+\infty}\lambda_k x^k$

example 4. Unit step function's series :

$\{Y_{x_i}\equiv Y_i\equiv Y(x-x_i);\ i=0,\ 1,\cdot\cdot\}$,

In this case, the approximation equals to piece-wise linear one : i.e.,

(2.3; 7) $F_{(5)}\triangleq\sum_{i\in N}\mu_i Y_i\cdot(x-x_i)+y_0$,

or,

(2.3; 7a) $=A_y\cdot x-B_y$,

where, A_y and B_y are definəd as follows :

(2.3; 8) $\begin{cases} A_y\triangleq\sum_{i\in N}\mu_i Y_i, \\ B_y\triangleq\sum_{i\in N}\mu_i Y_i+y_0, \end{cases}$

the unit step function Y_i is defined as follows:

(2.3; 9) $Y_i\equiv Y(x-x_i)=0$, for $x<x_i$,
$=1$, for $x>x_i$,

In the above linear functional space approximations, if the each norm is differentiable, the parameters $\alpha,\beta,\cdot\cdot$ are determined as follows :

(2.3; 10) $\min_{\lambda_k}\| f(x)+\sum_{k\in I}\lambda_k f_k\|=\min_{\lambda_k} N/\epsilon>$,

where, $I\subset N$, or, by differentiating the error norm $N/\epsilon>$:

(2.3; 11) $\dfrac{\partial}{\partial\lambda_k}N/\epsilon>=0,\ k\epsilon I$,

Until today, the almost approximating functions are linear, but in the near future, nonlinear approximating functions will be used frequently. Therefore next, nonlinear approximation will be mentioned.

2.4. Nonlinear Approximating Functions.

Nonlinear functions, concerning the parameters, are definid as follows.

(2.4; 1) $F(x;\lambda\alpha,\lambda\beta,\cdot\cdot)\neq\lambda F(x;\alpha,\beta,\cdot\cdot),\ \forall_\lambda$,

One example of the nonlinear functions is :

example 1. $\{P_i(x)e^{\alpha_i x};\ i\epsilon Z$(integer), and $P_i(x)$ is polynomial of any degree$\}$.

(2.4; 2) $F_{(6)}=\sum_i A_i P_i(x)e^{\alpha_i x}$,

Froms the point of view of the "Laplace trans form", the following formula :

(2.4; 3) $Y(t)\supset\dfrac{1}{P}$, $Y(t)\dfrac{t^{\alpha-1}}{\Gamma(\alpha)}\supset\dfrac{1}{P^\alpha}$,

$Y(t)e^{\lambda t}\dfrac{t^{\alpha-2}}{\Gamma(\alpha)}\supset\dfrac{1}{(P-\lambda)^\alpha}$,

are provided, then the difference between (2.4; 2) and (2.3; 6) (polynomial) are sligtly, but from the view point of approximation, these two methods are quite different, since the one is linear and the other is nonlinear.

2.5. Integral Approximation.

This approximation method may be seen the generalization of the above one, which contains some discrete parameters. The form of method

can be expressed as follows :

$$(2.5;1)\quad y=\int_a^b \Phi(x,\lambda)d\lambda,$$

The parameter λ, in this case, is completness and has finite measurable. One example of this method, i.e. "Laplace transform", is the extension of exponential functional approximation to the integral form :

$$(2.5;2)\quad y=\int_0^\infty F(\lambda)e^{-\lambda x}d\lambda,$$

As another example, Fourier transform may be aeso:

$$(2.5;3)\quad y(x)=\int_{-\infty}^{+\infty} F(\lambda)e^{2i\pi\lambda x}d\lambda,$$

or

$$(2.5;4)\quad F(\lambda)=\int_{-\infty}^{+\infty} y(x)\cdot e^{-2i\pi\lambda x}dx,$$

These approximation methods are very intereating mathematically, but practically they will become important in the future.

2. 6. example

2. 6. 1. example 1. curve-fitting or point-approximation problem.

As one example, let us consider the following curve-fitting problem of Fig. 1.

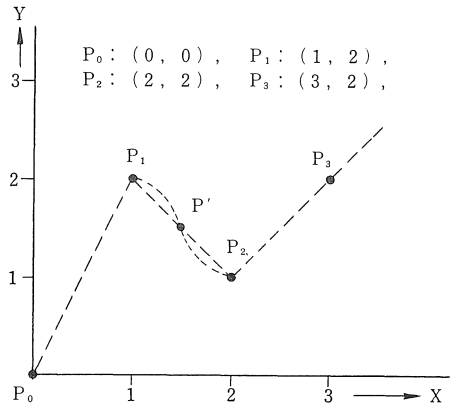

Fig. 1 The example of curve fitting problem.

In this case, D_0 in the (2.1; 2) is that: $(x_1^{(0)}, y_1^{(0)})=(0,0)$, $(x_2^{(0)}, y_2^{(0)})=(1,2)$, $(x_3^{(0)}, y_3^{(0)})=(2,1)$, $(x_4^{(0)}, y_4^{(0)})=(3,2)$.

2. 6. 1. 1. Polynomial Approximation

If this curve is approximated by three degree equation :

$$(2.6.1;1)\quad y=s_0+s_1x+s_2x^2+s_3x^3$$

And using the data of $y_i^{(0)}=y(x_i^{(0)})$, the coefficients $(s_k)_{k\in\{0,1,2,3\}}$ may be calculated asfollows :

$$(2.6.1;2)\quad (s_0,s_1,s_2,s_3)=(0, 5.17, -14, 0.83)$$

Next, considering the information of D_1, i.e. $(dy/dx)_{at(1,2)}=0$, the coefficients s_k become as follows :

$$(2.6.1;3)\quad (s_0,s_1,s_2,s_3)=(0, 4.5, +3, 0.5)$$

And further, assuming the following information of D_2, i.e., $(d^2y/dx^2)_{at(1.5,1.5)}=0$, the coefficients s_k are as follows :

$$(2.6.1;4)\quad (s_0,s_1,s_2,s_3)=(0, 5.5, -4.5, 1),$$

The geometric diagram of the curve of equation (2.6.1; 1), using these values of coefficients, are plotted in Fig. 2(a) and (b).

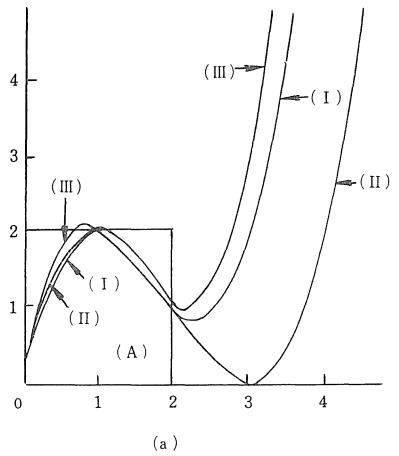

Fig. 2(a) The Approximation Curve, (Ⅰ), (Ⅱ) (Ⅲ)

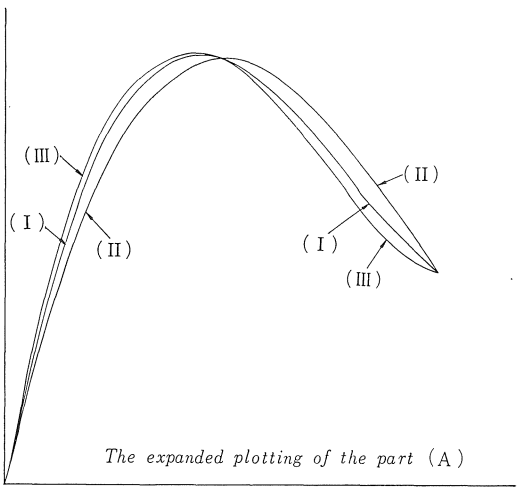

Fig. 2(b). The Approximation Curve of (Ⅰ), (Ⅱ), (Ⅲ).

And if this curve may be expressed piecewise linear approximation, using unit step function of (2.3; 7), the coefficients $(\mu i)_{i\in N}$, ui and Y_i become as follows :

(2.6.1; 5) $y(x)=2Y(x)\cdot x-3Y(x-1)\cdot(x-1)+$
$+2Y(x-2)\cdot(x-2)$,

or,

(2.6.1; 5a) $y(x)=(2Y_0-3Y_1+2Y_2)\cdot x$
$-(-3Y_1+4Y_2)$,

these expressions will be used in the equivalent-linearization method soon later.

2.6.2. Example 2. Plate current - Grid voltage Characteristics of a triod.

In this case, also, the three forms of expressions may be possible, that is, polynomial (or Taylor series), Fourier - series representation, and piece-wise linear method. [see reference [4]. However, only two representation will be considered here. First, consider the Fourier-series representation which is pictured in Fig. 3a.

If y represents the current and x the voltage, then owing to the fact that the characteristic

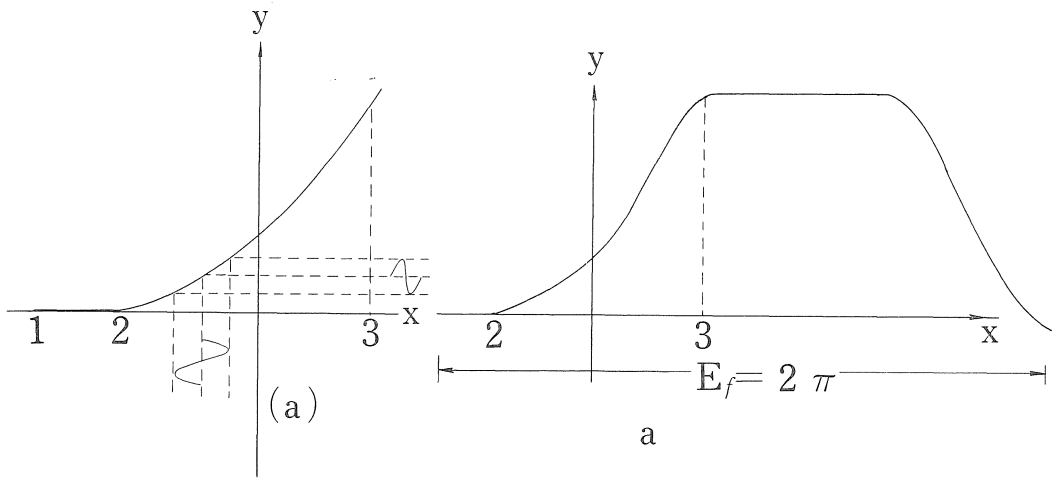

Fig. 3.

curve in Fig. 3b is even, it follows that

(2.6.2; 1) $y=\frac{a_0}{2}+\sum_{n=1}^{N} a_n\cos\left(\left(\frac{2\pi n}{E_f}\right)x+\Theta_n\right)$,

Using this expresion, following the reference [4], if the grid voltage is written as follows:

(2.6.2; 2) $x=E_0+E_1\sin\omega_1 t$

It then follows that

(2.6.2; 3) $y=\frac{a_0}{2}+\sum_{n=1}^{N} a_n\cos(u_n+v_n\sin\omega_1 t)$

where

(2.6.2; 4) $u_n=\frac{2\pi n}{E_f}E_0+\Theta_n$

$v_n=\frac{2\pi n}{E_f}E_1$

equation (2.6.2; 3) may be expanded into the form,

(2.6.2; 5) $y=\frac{a_0}{2}+\sum_{i=1}^{N} a_n[\cos u_n\cos(v_n\sin\omega_1 t)-$
$-\sin u_n\sin(v_n\sin\omega_1 t)]$,

the mth harmonic of the plat current is written by the expression,

(2.6.2; 6) $|x_m|=2\sum_{n=1}^{N} a_n J_m(v_n)\cos\left(u_n+\frac{m\pi}{2}\right)$,

The above characteristic of triod may be expressed by the piece wise linear method, as the same as the case of (2.6.1).

In the above all cases, the suffix set I or N may be all integer z, or all rational nunaber Q, or all real number R or other set. [Remark] In linear approximation, if the curve is the v-i characteristics of 2-terminal elements, then it may be represented as follows:

(A. 1) $i_D=a_0f_0(v)+a_1f_1(v)+\cdots=\sum_{k\in N}a_kf_k(v)$,

this expression has a important meaning on the circiut topology, that is, the each term a_kf_k provides the shunt circuit as shown in Fig. 4. From this view point, the relation between the nonlinear mathematics and the circuit topolgy will be clarify as gradually as oc

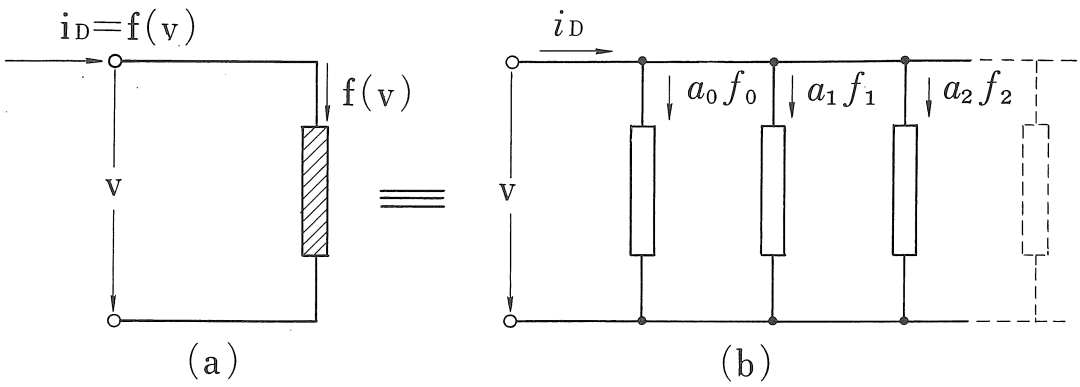

Fig. 4. (b) is the equivalent cijcuit of (a).

3. Generalization of Equivalent Linearization and Describing Function Method

3. 1. Nonlinear Operator

The Nonlinear operator is defined as follows:

(3.1; 1) $H(x)\triangleq H(x,F_{(i)}(x),D_t)(x)$

where, assuming as follows:

(3.1; 2) $\begin{cases} D_t\triangleq d/dt; \\ F_{(i)}(x)\triangleq \text{ polynomial or analytic function of } x; \end{cases}$

and, H is a polynomial, or analytic function of x, $F_{(i)}$, and D_t. Let us denote: $x\equiv\xi_1$, $F_{(i)}\equiv\xi_2$, and $D_t\equiv\xi_3$, then H may be considered as the function of three variable, ξ_1, ξ_2, ξ_3. And further let us write as follows:

(3.1; 3) $\xi_1^{P_1}\xi_2^{P_2}\xi_3^{P_3}\equiv\xi^P$, P_1, P_2, P_3 are integer.

$$|p|=\sum_{i=1}^{3} p_i,$$

then, if the degree of polynomial H is smaller than m, the function H can be written as follows:

(3.1; 4) $H(\xi_1,\xi_2,\xi_3)=\sum_{|P|\leq m} a_p\xi^p,$

But in this case as different from the general polynomials, it must be noted that, algebra is noncommutative. This is very important point. Generally, in nonlinear mathematics, the algebra, to which the mathematics belongs, is the noncommutative one. From this viewpoint, it may be thought that "nonlinearity" was very closely related to knot theory. Particullary, a limit cycle or a periodical oscillation will be studied from these directions in the future.

Now, let us return to the equation (3.1; 1), and consider the particular example of nonlinear operator H.

example 1. Van der Pol operator:

(3.1; 5) $y\triangleq H(x)\triangleq D_t^2x+\mu(x^2-1)D_tx+x,$

example 2. Duffing operator:

(3.1; 6) $y\triangleq H(x)\triangleq D_t^2x+x+\mu x^3,$

example 3. One retarded action operation:

(3.1; 7) $y\triangleq D_t^2x+2(\delta+\delta_2x^2)D_tx+\nu_0^2x+\nu^2x_\tau,$

where δ, δ_2, ν_0^2, and ν^2 are constants, and x_τ is defined as the form:

$x_\tau\triangleq x(t-\tau)$, τ is retarded time,

These operators have been well known, but rather classical. For this reason, they are very typical examples, in the case of studying the generalized equivalent linearization procedure.

3. 2. Systems of Nonlinear Differential Equations.

The systems of nonlinear differential equations are given as the following forms:

(3.2; 1) $M\boldsymbol{x}=\boldsymbol{f}(\boldsymbol{x}),$

where, M is the n row-n column matrix, $\boldsymbol{x}$ is the vector $\boldsymbol{x}=(x_1,x_2,\cdots,x_n)^t$, and $\dot{x}$ is denoted the derivative with respect to time t. $\boldsymbol{f}$ is, also, a vector, of which component f_i consists of the functions of variable $x_1,\cdot\cdot,\ x_n$, $i=1,\cdot\cdot,n$, respectively. Here, we shall consider two cases respectively, that is:

Case 1. $\det M\neq 0$. In this cases, equation (3.2; 1) may be rewritten as the following form:

(3.2; 2) $\Omega\cdot\boldsymbol{x}=\boldsymbol{f}(\boldsymbol{x})$ or, $I\cdot\boldsymbol{x}=\boldsymbol{f}(\boldsymbol{x}),$

where $\Omega=\mathrm{diag}[\alpha_1,\cdot\cdot,\alpha_n]$, and I is unit matrix: $I=\mathrm{diag}\,[1,\cdot\cdot,1]$, assuming that every number α_i is not equals to zero. In the equation (3.2; 2),

the independent variable t is not contained explicitly. But, in the case of containing the varioble t explicitly, and if we denote $t \triangleq x_{n+1}$, and $dx_{n+1}/dt = 1 \triangleq f_{n+1}(x_1, \cdots, x_{n+1})$ respectively, the equation of the case equals to the systems of $n+1$ differential equations of n+1 variables, formally.

Case 2. det $M=0$, In this case, equation (3.2; 1) is degenerated, that is, the diagonal matirx Ω is expressed as the form,

$$\Omega = \text{diag}\,[\alpha_1, \cdots, \alpha_r, 0, \cdots, 0],$$

Then (3.2; 1) may be divided into two groups: r differential equations and the $(n-r)$ algebraric equations. The system, at that case, generats pulse, and the such oscillator is called the relaxation oscillator. The differential equations have mathematically become the singular perturbation problem. and therefore analytical solution can be hardly obtained by the ordinary method. Then, we have omitted this case.

Equation of case (3.1) can be always tranformed to the the system (3.2). For example, Van der Pol operator (3.1; 5) may be reduced to the systems as follows:

Let us put, $x \equiv x_1$ and $D_t x, \equiv x_2$,
then, equation (3.1; 5) is :

$$(3.2;\ 3)\quad \begin{cases} D_t x_1 = x_2 \\ D_t x_2 = -x_1 + \mu(1-x_1{}^2)x_2, \end{cases}$$

in this case, the functions f_1, f_2 becom into the forms :

$$(3.2;\ 3a)\quad \begin{cases} f_1(x_1, x_2) \triangleq x_2, \\ f_2(x_1 x_2) \triangleq -x_1 + \mu(1-x_1^2)x_2, \end{cases}$$

However, Van der Pol operator as defined (3.1; 5) is much convenient for practial applications very often And for that reason, we have divided into the two cases, i.e., the one is operator (3.1), the other is systems (3.2).

3. 3. The Generalization of Equivalent Linearization.

Equivalent Linarization are provided from the many different point of view, such as, tangent linearization, optimal linearization, and so on. But, all-of the above formulation, we may conclude that as follows : "The equivalent linearization, is, the one of taking the mean value, or the balancing <in some sense>". That is, in the case (3.1), if the operator H^* is defined as follows:

$$(3.3;\ 1)\quad H^* \triangle \frac{<H(x)>}{<x>}$$

$$\text{or} \quad \triangle \frac{<H(x),\ \varphi_1(x)>}{<x,\ \varphi_1(x)>},$$

then, H^* may be considered as the operator, which operates to x, in place of operator H: that is,

$$(3.3;\ 2)\quad y = H(x) \Longrightarrow y = H^* \cdot x,$$

In other words, operator H is deduced to H^* the operator H^* has, now, be-come linear. the latter in (3.3; 1) is the weighted mean value, and this form, is just equals to the Schwartz's distributions, assuming that $H(x) \in D'$, $x \in D'$, and $\varphi \in D$.

However, at the present case, some questions must be answered for justifying the above formula (3.3; 1):

Question 1.

The method for producing the linear operator associated with a given nonlinear operator requires knowledge of the function $x = x_0(t)$, the point of the functional space, at which the linearization is performed.

Question 2.

What method of taking the meanvalue or balancing could be recomended in each case ? This problem may be, justly reduced to the norm or the distance of the given functional space.

Question 3.

Once the functional space, and its norm or the distance has been sepecified, what functions may be selected as $\varphi(x)$?

Once these
above questions have been answered,
H^* could be determined. Next we shall considered some practical examples.

Example 1. If we select the Hilbert space as functional space, and L_2 norm as the distance in the space, then, in the equation (3.3; 1),

$$(3.3;\ 3)\quad \begin{cases} \varphi_1(x) \equiv H(x), \\ \varphi_2(x) \equiv x, \end{cases}$$

and, $\quad <f(x),\ \varphi(x)> \equiv \int_a^b f(x)\varphi(x)d(x)$

can be put, and, let us put $x_0 = a_0 \cos \omega t$, the point in the space.

Then, the next equation may be hold:

$$(3.3;\ 4)\quad H^* = \frac{\int_a^b (H(a_0 \cos \omega t))^2 dt}{\int_a^b a_0{}^2 \cos^2 \omega t\ dt,}$$

In the case of Van der Pol operator the above formula is:

$$(3.3;5)\quad H^*=D_t{}^2+\mu\left(\frac{a_0{}^2}{4}-1\right)D_t+1,$$

As conclusion, the above equation (3.3; 1) holds in general, that is, the equation is the complete generalization of equivalent linearization in the case of nonlinear operator.

3. 3. 1. The extension to the Systems of Differential Equations.

Let us consider, now, that the systems of differential equation are given: $\dot{x}=f(x)$. In this case one question occurs; that is, what point in the functional space must be selected?

$$(3.3;6)\quad x_0=(x_{10}, x_{20}, \cdots x_{n0})^t,$$

In general, this question is very difficult, but, if the solution has the oscillatory property, the following assumption may be seen as justifying:

$$(3.3;6a)\quad \begin{cases} x_{10}=a_1\cos(\omega_1 t+\varphi_1), \\ x_{i0}=a_i\cos(\omega_i t+\varphi_i), \end{cases}$$
$$(i=1\cdots, n)$$

And further all of the above case, assuming the following fact:

$$(3.3;6b)\quad \omega_i=\omega,\ \forall i\in N=\{1, \cdots, n\}$$

This assumption may be justified as follows: if $\omega_1=2\omega_2$, and $n=2$, then the limit cycle diagram in x_1-x_2 plane, will be come as the Fig. 5a.

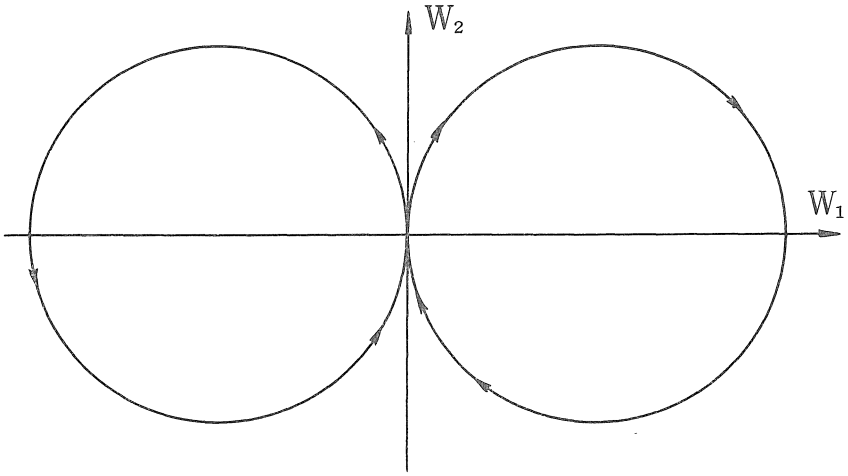

Fig. 5(a) The limit cycle of oscillation in the case: $w_1=2w_2$.

However, in Fig. 5, singular point, the origin P_0, is satisfied by the differential equation. This means that at the point (0, 0), $f_1(0, 0)$ and $f_2(0,0)$ have not the specified value.

Such case may be occur very suddenly. Therefore, usally, it is natual that the assumption: $\omega_i\cong\omega_j$, $(i\neq j)$ $\forall i, j$, may be hold in almost cases. Here after, we shall think so. Then, assuming the following equation:

$$\dot{x}_i=\dot{a}_i\cos(\omega t+\varphi_i)+\cdots\cdot,$$

tangent linearization method is that:

$$(3.3;7)\quad \dot{x}_i=\frac{\dot{a}_i}{a_i}x_i+\cdots,$$
$$(i=1, \cdots, n)$$

All other cases may be reduced from the formula (3,3; 1). If $x=x_0(t)$ is Fourier series; $x_{0j}(t)=\sum_{k=-\infty}^{+\infty}C_k{}^{(j)}e^{ikwt}$, then, $\varphi(x)$ will become e^{-ikwt}, and the $\langle,\rangle$ equals $\int_0^T H\cdot e^{-ikwt}\frac{dt}{T}$.

Moreover, H^* will be changed to a vector; that is,

$$(3.3;8)\quad H^{*(j)}=\left(\frac{\langle H(\sum C_k(j)e^{ikwt}),\ e^{-iwt}\rangle}{\langle\sum C_k e^{ikwt},\ e^{-iwt}\rangle},\ \frac{\langle H(\sum C_k{}^{(j)}\cdots),\ e^{-i2wt}\rangle}{\langle\sum C_k{}^{(j)}\cdots,\ e^{-i2w_1}\rangle},\cdots\right)$$

From the above example, we can understand that, "the equivalent linearization is equalent to ≪taking-mean value method≫ in some sense, or in a meaning".

3. 4. The generalization of Van der Pol and Krylov-Bogoliubov Method.

KJ.B. method is equal to transform the given (x, y) coordincte on R^2 to the polar coordinate (r, θ) on the same R^2 plane, taking the mean value with respect to the angle θ.

This method may be extended to the R^n space,

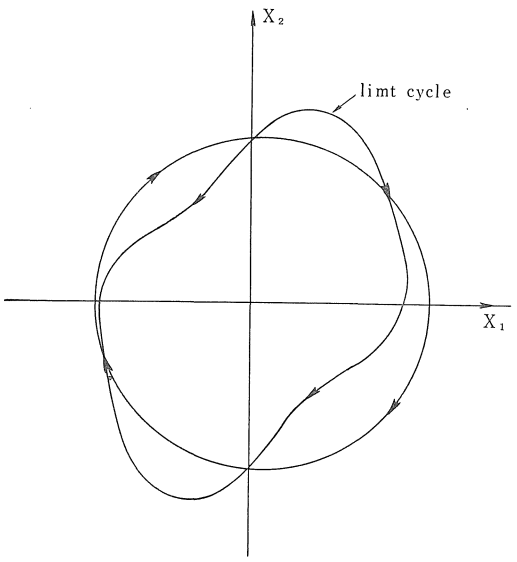

Fig. 6. The limit cycles of Van der Pol oscillator: $\dot{x}_1=x_2$, $\dot{x}_2=-x_1+\mu(1-x^2)x_2$ and. its eqiuvalent, optimal linearized oscillator.

adapting the generalized polar coordinate on R^n, that is:

$$(3.4;\ 1)\quad x_i=r\sin\theta_{n+1-i}\prod_{j=1}^{n-i}\cos\theta_j \quad \text{with } \theta_n=\frac{\pi}{2} \qquad (i=1,\cdot\cdot,n)$$

or, in other form,

$$(3.4;\ 1a)\quad x_i=r\sin\theta_{n+1-i}\prod_{j=1}^{n-i}\sin\theta_j, \text{ with } \theta_n=0 \qquad (i=1,\cdot\cdot,u)$$

The different point between (3.4; 1) and (3.4; 1a) is the direction of the R^n space[1], and this will be not important for the application of the trans formation of coordinate.

Consider, that the system $\dot{x}=f(x)$ are given. and change the variable x to the polar coordinate, which is given in (3.4; 1). Then, the form of the system will be trans formed to the following form: i.e.

$$(3.4;\ 2)\quad \begin{cases} \dot{r}\sin\theta_n\prod_{j}^{n-1}\cos\theta_j-\dot{\theta}_1 r\sin\theta_n+\cdots \\ \qquad\qquad =f_1(r\sin\theta_n\cdots), \\ \dot{r}\sin\theta_{n-1}\prod_{j}^{n-2}\sin\theta_j-\dot{\theta}_1 r\cdots+\cdots \\ \qquad\qquad =f_2(r\sin\theta_n,\cdots), \\ \cdots\cdots\cdots \\ \dot{r}\sin\theta+\cdot\theta_1 r\cdots+\cdots=f_n(r\sin\theta_n), \end{cases}$$

1) it may be concluded from knot they; for examples, see the book: "introduction to knot theory" by R. Hcroowell and Ralph H. Fox.

The left sides of these equations are the linear equation of n variables, $\dot{r}$, $\dot{\theta}_1$, $\dot{\theta}_2\cdots$, $\dot{\theta}_{n-1}$, and therefore, (3.4; 2) can be transformed to the equation of $\dot{r}$, $\dot{\theta}_1,\cdots$. Then, these equations will be reduced to the following system:

$$(3.4;\ 3)\quad \begin{cases} \dot{r}=F_r(r,\ \theta_1,\cdots,\ \theta_{n-1}), \\ \dot{\theta}_1=F_{\theta 1}(r,\ \theta_1,\cdots,\ \theta_{n-1}), \\ \cdots\cdots\cdots \\ \dot{\theta}_{n-1}=F_{\theta n-1}(r,\ \theta_1,\cdots,\ \theta_{n-1}), \end{cases}$$

and, if we may take the mean value the equations on right sides on the R^n sphere $r^{n-1}S_n$ (where S_n is the area of the radious 1 onR^n), the right hand side might be reduced to the functions of only one variable r, that is:

$$(3.4;\ 4)\quad \begin{cases} \dfrac{1}{r^{n-1}S_n}\displaystyle\int F_r(r,\theta_1,\cdots,\theta_n)r^{n-1}dS_n\equiv F_r(r), \\ \dfrac{1}{r^{n-1}S_n}\displaystyle\int F_{\theta i}(r,\theta_1,\cdots,\theta_n)r^{n-1}dS_n\equiv F_{\theta i}(r), \\ \qquad (i=1,\ \cdots,\ n-1) \end{cases}$$

and,

$$(3.4;\ 3a)\quad \begin{cases} \dot{r}=F_r(r), \\ \dot{\theta}_1=F_{\theta 1}(r), \\ \cdots\cdots\cdots \\ \dot{\theta}_{n-1}=F_{\theta n-1}(r), \end{cases}$$

The equations on (3.4; 3a) have important properties for practical applications, since, at first' within these equation being considered, angle coordinates, $\theta_1,\cdots$ are all the function of variable r only, and from the first equation on (3.4; 3a), the oscillation concerning the nonlinear equations may be concluded whether the oscillation stable or unstable one. Coneer.-ning with to respect the point, a usual equivalent liniearization method will provid nothing. Now, on the fol lowing, it willde mention ed some particular examples.

Example 1. the case $n=2$.

Consider the following system of differential equations:

$$(3.4;\ 4)\quad \begin{cases} \dot{x}_1=f_1(x_1,\ x_2), \\ \dot{x}_2=f_2(x_1,\ x_2), \end{cases}$$

and, transform the $(x_1,\ x_2)$ to the $(r,\ \theta_1)$:

$$(3.4;\ 5)\quad \begin{cases} x_1=r\cos\theta, \\ x_2=r\sin\theta, \end{cases}$$

(this can be reduced from (3.4; 1)).

Then, substituting in the (3.4; 4) ;

(3.4; 6) $\dot{r}\cos\theta_1+\dot{\theta}_1 r(-\sin\theta_1)=f_1(r\cos\theta_1, r\sin\theta_1)$
$\dot{r}\sin\theta_1+\dot{\theta}_1 r\cos\theta_1=f_2(r\cos\theta_1, r\sin\theta_1).$

from the above equations, with respect to the variable r, θ_1, the following equation may be obtained:

(3.4; 7) $$\begin{cases}\dot{r}=f_1\cos\theta_1+f_2\sin\theta_1,\\ r\dot{\theta}_1=f_2\cos\theta_1-f_1\sin\theta_1,\end{cases}$$

these equations correspond to (3.4; 3).

The calculationg corresponding to (3.4; 4) with respect to θ_1, have been performed, the next formula will result:

(3.4; 8) $$\begin{cases}\dot{r}=F_r(r),\\ r\dot{\theta}_1=F_{\theta 1}(r),\end{cases}$$

where,

(3.4; 8a) $$\begin{cases}F_r(r)\triangleq\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}\{f_1\cdot\cos S+f_2\cdot\sin S\}dS,\\ F_{\theta 1}(r)\triangleq\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}\{f_2\cos S-f_1\cdot\sin S\}dS,\end{cases}$$

As the particular example, consider Van der Pol operator: $f_1\triangleq x_2$, $f_2\triangleq -x_1+\mu(1-x_1{}^2)x_2$, (see, the equation (3.2; 3)). Let us calculate $F_r(r)$, and $F_{\theta 1}(r)$:

$$\begin{aligned}F_r(r)&=\frac{1}{2\pi}\int_0^{2\pi}\{r\sin S\cos S+(-r\cos S+\mu\\ &\qquad(1-r^2\cos^2 S)r\sin S)\sin S\}\cdot dS\\ &=\frac{r}{2\pi}\int_0^{2\pi}\{-\cos S\sin S+\mu\sin^2 S-\\ &\qquad-\mu r^2\cos^2 S\sin^2 S\}dS\\ &=\frac{r}{2\pi}\left\{\frac{\mu}{2}\cdot 2\pi-\frac{\mu r^2}{8}2\pi\right\}\\ &=\frac{\mu r}{8}(4-r^2),\end{aligned}$$

and,

$$\begin{aligned}F_{\theta 1}(r)&=\frac{1}{2\pi}\int_0^{2\pi}\{-r\cos S+\mu(1-r^2\cos^2 S)r\sin S)\\ &\qquad\cos S-r\sin S\sin S\}\cdot dS\\ &=\frac{1}{2\pi}\int_0^{2\pi}r\{-1-\mu r^2\cos^3 S\sin S\}dS\\ &=\frac{-r}{2\pi}\int_0^{2\pi}(1+\mu r^2\cos^3 S\sin S)dS\\ &=-r,\end{aligned}$$

Then, from (3.4; 8),

(3.4; 8a) $$\begin{cases}\dot{r}=\dfrac{\mu r}{8}(4-r^2),\\ \dot{\theta}=-1,\end{cases}$$

if the initial condition will be $r=r_0$, $\theta_0=0$ at $t=0$, the solutions of (3.4; 8a) are:

$$\frac{r^2(2-r)}{2+r}=Ae^{-\mu t},\quad \theta_1=-t,$$

Then, the steady state amplitude is $r_s=2$, and the fundamental angular velocity is equals to 1. And moreover, it may be known that the transient time until the steady state motion is propontional to the value μ.

As a concluding remark, if generalized B.- K. method will be developed to the system of the differential equations more completely, particularly in relation to knot theory (i.e.topological geometry), this method will be come the most powerfull tool for analizing the nonlinear differential -difference systems.

3. 5. Describing Function Method

Describing function method, strongly depends on the properties of complex numbers. Complex numbers have been the significant tool for analizing the electronic com munication circuits until tody, particularly linear circuit. For that reason, describing function method has been studied in details, and particularlly it was used very often. relating nonlinear control theory,

Now, however, what character of complex number will be related to the deseribing function method, algebraic or analytic? It will be thought for me that, "some ortho gonal properties" are activity in this case. If it is true, other ortho gonal functions must be used at the same way, in place of complex. This is the fundamental idea of the present paper. And further the describing function method are very strongly related to the stability of the system. About this point, we must note at the same time.

If the system can be represented by one nonlinear operator H, the coresponding des cribing function can be expressed as the following form:

(3.5; 1) $$H\triangleq\frac{<H(a_0\cos\omega t),\ e^{-i\omega t}>}{<a_0\cos\omega t,\ e^{-i\omega t}>}$$

or,

$$=\frac{\displaystyle\int_0^{2\pi/\omega}[H(a_0\cos\omega t)]e^{-i\omega t}dt}{\displaystyle\int_0^{2\pi/\omega}a_0\cos\omega t e^{-i\omega t}dt}$$

or,

$$=\frac{\omega}{\pi a_0}\int_0^{2\pi/\omega}[H(a_0\cos\omega t)]e^{-i\omega t}dt,$$

Then, the describing function is equivalent to the equivalent linearization, concerning $e^{-i\omega t} \in D(\Gamma)$ in Hilbert space. This is very important thing However, the above formula, used in definition shows that the ordinary describing function can be extended to the more various directions. For example, if $k\omega$ could be replaced to ω, in (3.5; 1), the following formula may be defined:

(3.5; 2) $$H_k \triangleq \frac{<H(a_0\cos k\omega t),\ e^{-ik\omega t}>}{<a_0\cos k\omega t,\ e^{-ik\omega t}>},$$

This is the describing function of the kth order harmonics. More generally, the following formula may be considered:

(3.5; 3) $$H^* \triangleq \frac{<H(x_0),\ \varphi_0(t)>}{<x_0,\ \varphi_0(t)>},$$

The above equation defines the generalized describing function.

And further, in the (3.5; 3), H^* is scalar, but H^* may be extended to vector, or matrix. But, this will be future problem, and here these fact are not considered.

〔Remark 1〕 Dual Network, or Dual system:

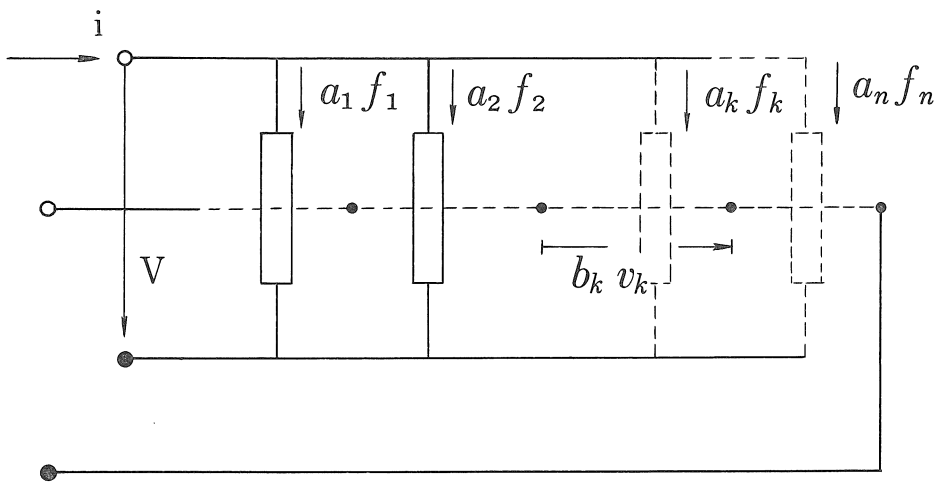

Fig. 7. Nonlinear two terminal network and its dual network

Nonlinear dual network, may be considered, of course. As shown in Fig. 6. and, let us Consider analytical expresion as following form:

(R1; 1) $i=\sum_k a_k i_k(v)$,

(R1; 2) $v_d=\sum_j b_k v_k(i_d)$,

If these two dual networks have the same energy, the next equation hold:

(R1; 3) $(i,\ \sum_k a_k i_k(v))=(v_d,\ \sum_j b_j v_f(i_d))$,

i.e.,

(R1; 3a) $\sum_k a_k(i,\ i_k(v))=\sum_j b_j(v_d,\ vd(i_d))$

And if the each $(i,\ i_k(v))$ equals to the corresponding $(v_d,\ v_j(i_d))$, the following formula will be hold:

(R1; 4) $$\begin{cases} a_k=b_k, & \text{and} \\ (i,\ i_k(v))=(v_d,\ v_k(i_d)). & \\ \qquad \text{for any } k, & \end{cases}$$

This is the generalized orthogonal condition. And considering the relation between nonlinear network and its mathmatical expression will be very important

〔Remark 2〕 Piece-wise Linear Approximations:

Piece-wise linear approximation method contains unit step functions in the equation. Therefore his derivatives have impulse (delta) function and its derivatives; that is L. Schwartz's distributions. But, trigger voltage in Multivibrator is already equals to the δ function, mathematically. And, in this respect to the point, there are no difficult problem. Rather their practical utilities are very important, that is, by using the piecewase linear approximation, very sigificant character of nonlinier circuit will be appear, (omitting in this paper), and δ function as the forcing function is very useful function in the Van Slooten's theory.[1)]

4. Conclusion.

The General theory and the important points

of functional approximations of various charac teristic curves of electronic elemennts and generalization of equivalent linearization and describing functions were explained and discussed in this paper.

The theory and the methads given in this paper are supposed to be very useful for analyzing the nonlinear circuit problems. The practical applications of this theory to many interesting examples will be reported in the near future.

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References

〔1〕 Thomas. E. Stern: "Theory of Nonlinear Networks and Systems" 1965. Addison - wesley. Reading massa chusetts

〔2〕 Laurent Schwartz: "Méthodes Mathématiques pour Les Sciences Physiques". 1961, Editions Scientifiques Hermann, Paris.

〔3〕 新美："負性抵抗の種々な特性近似による発振特性の解析" Japan, electronic communication society, 1967, June, 15th, 非直線理論研究会資料.

〔4〕 Austin Blaquiére: "Nonlinear System Analysis" 1966, Academic Press.

〔5〕 C. Louis Cuccia: "Harmonics, Sidebands, and Tansients in Communication Engineering".